Real Analysis Qualifying Exam, Fall 2006
(2 hours)

Instructions: Work problems 1-3 and any 3 of the other 4 problems.

#1. (15 points) Let B be a bounded subset of $\mathbb{R}^n$. Prove the following:
a) If B is closed in $\mathbb{R}^n$ and $f : B \to \mathbb{R}$ is a continuous function, then f is uniformly continuous.
b) If B is not closed in $\mathbb{R}^n$, then there exists a continuous $f : B \to \mathbb{R}$ which is not uniformly continuous.

#2. (10 points) Show that the sequence of functions $f_n = n^{-1}\chi_{(0,n)}$ converges uniformly to the 0-function on $\mathbb{R}$ but does not converge to 0 in L^1. Here $n = 1, 2, \ldots$ and $\chi_{(0,n)}$ denotes the characteristic function of $(0, n) \subset \mathbb{R}$.

#3. (15 points) Let $\{f_k\}_{k=1}^{\infty}$ be a uniformly bounded sequence of functions on $[0, 1]$ which converges pointwise to a function f. Answer the following with a brief explanation or a counterexample:
a) If each f_k is Lebesgue integrable, does it follow that f is Lebesgue integrable?
b) If each f_k is Riemann integrable, does it follow that f is Riemann integrable?

#4. (20 points) Let m be Lebesgue measure on $\mathbb{R}^d$, and let $\{f_k\}_{k=1}^{\infty}$ be a sequence of measurable functions defined on a measurable set E with $m(E) < \infty$ such that $f_k \to f$ pointwise on E. Prove that for any $\epsilon > 0$ there exists a subset A of E such that $m(E - A) \leq \epsilon$ and $f_k \to f$ uniformly on A.

#5. (20 points) Define the norm on $L^2(\mathbb{R})$ and give an argument for the completeness of $L^2(\mathbb{R})$.

#6. (20 points)
a) Show that for $f \in L^1(\mathbb{R}^n)$, if $\hat{f}$ is defined on $\mathbb{R}^n$ by $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i x\cdot\xi}dx$, then $\hat{f}$ is a continuous function.
b) Indicate how to obtain an example of $f \in L^1(\mathbb{R})$, such that $\hat{f}$ is not a differentiable function. [No justification required.]

#7. (20 points) If m is Lebesgue measure on $\mathbb{R}$ and $f \in L^1(\mathbb{R})$, let $F(x) = \int_{(-\infty,x]} f\,dm$, for $x \in \mathbb{R}$. Show that for almost every x, $F'(x)$ exists and is equal to $f(x)$. [Give the main points of a proof and indicate some of the details.]