Real Analysis Qualifying Examination

August 2009

There are five problems, each worth 20 points. Give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Let $a > 1$ and $f_n(x) = \dfrac{x}{1+x^n}$ for each $n \in \mathbb{N}$ and $x \geq 1$.

   (a) Show that $\sum_{n=1}^{\infty} f_n$ converges uniformly to a continuously differentiable sum function s on $[a, \infty)$ and that $s'(x) = \sum_{n=1}^{\infty} f_n'(x)$ for all $x \in [a, \infty)$.

   (b) Show <u>directly from the definition</u> that f_n converges uniformly to 0 on $[a, \infty)$, but does <u>not</u> converge <u>uniformly to 0 on $(1, \infty)$</u>.

2. (a) Prove that $L^3([0, 2\pi]) \subseteq L^2([0, 2\pi])$ and

$$\left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx \right)^{1/2} \leq \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^3 \, dx \right)^{1/3}.$$

   (b) Prove that $\ell^2(\mathbb{Z}) \subseteq \ell^3(\mathbb{Z})$ and

$$\left(\sum_{k=1}^{\infty} |a_k|^3 \right)^{1/3} \leq \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2}.$$

3. Let $f \in L^1([0, 1])$.

   (a) Show that for any $\epsilon > 0$ we can write $f = g + h$, where $g \in L^2$ and $\|h\|_1 < \epsilon$.

   (b) Use this decomposition (or other means) to prove that

$$\lim_{k \to \infty} \int_0^1 f(x) e^{-2\pi i k x} \, dx = 0.$$

4. Let $f \in L^1([0, 1])$, $\{f_k\}_{k=1}^{\infty}$ be a sequence of Lebesgue square-integrable functions with $\|f_k\|_2 \leq 1$ for all k, and suppose that

$$\lim_{k \to \infty} \int_0^1 |f_k(x) - f(x)| \, dx = 0.$$

   (a) Prove that $f \in L^2([0, 1])$ and $\|f\|_2 \leq 1$.

   (b) Do the above hypotheses guarantee that f_k will converge to f in $L^2([0, 1])$, namely that

$$\lim_{k \to \infty} \int_0^1 |f_k(x) - f(x)|^2 \, dx = 0 \ ?$$

   Be sure to justify your answer.

5. Let E be a Lebesgue measurable subset of $\mathbb{R}$.

   (a) Suppose 0 is a point of Lebesgue density of E. Show that there is an infinite sequence of points $\{x_k\}_{k=1}^{\infty}$, with $x_k \neq 0$ and $x_k \to 0$, such that $\{-x_k, x_k\} \subseteq E$ for all k.

   [Recall that x is said to be a *point of Lebesgue density of* E if $\displaystyle\lim_{h \to 0} \frac{m(E \cap (x-h, x+h))}{2h} = 1.$]

   (b) Prove that for almost every $x \in E$, there is an infinite sequence of points $\{x_k\}_{k=1}^{\infty}$, with $x_k \neq 0$ and $x_k \to 0$, such that $\{x - x_k, x, x + x_k\} \subseteq E$ for all k.