Probo

Ph.D. Prelim: PROBABILITY THEORY — Spring '95

1. (a) Quote Doob's martingale convergence theorem.

   (b) Define a sequence $\{X_n, n \geq 0\}$ of random variables as follows: $X_0 = 0$; if, for $n \geq 1$, $X_{n-1} = 0$, define $X_n = 1, -1, 0$ with probabilities $\frac{1}{2n}, \frac{1}{2n}, 1 - \frac{1}{n}$, respectively; if $X_{n-1} \neq 0$, define $X_n = nX_{n-1}, 0$ with probabilities $\frac{1}{n}, 1 - \frac{1}{n}$ respectively. Show that (i) $\{X_n\}$ is a martingale, (ii) $X_n \to 0$ in probability, and (iii) X_n does not converge to 0 almost surely.

2. (a) Quote (without proof) Kolmogorov's three series theorem.

   (b) If $\{X_n\}$ is a sequence of independent random variables with zero mean and satisfying

$$\sum_{n=1}^{\infty} E\left[X_n^2 1_{\{|X_n| \leq 1\}} + |X_n| 1_{\{|X_n| > 1\}}\right] < \infty,$$

   show that $\sum_{n=1}^{\infty} X_n$ converges almost surely.

3. (a) Quote a criteria (necessary and sufficient conditions) for the precompactness of a family of probability measures on the space of continuous functions on $[0, 1]$.

   (b) Show that $X_n \Rightarrow 0$ if and only if $X_n \to 0$ in probability.

4. (a) If the random variables $X_1, \ldots, X_n$ are independent with $E[|X_k|] < \infty$, $\forall\, k$, show that

$$E\left\{\prod_{k=1}^{n} X_k\right\} = \prod_{k=1}^{n} E[X_k].$$

   (b) Let X and Y be independent random variables with distribution functions $F(x)$ and $G(y)$ respectively. Show that

$$P(X + Y \leq z) = \int F(z - y)\, dG(y).$$

5. (a) Quote (without proof) the First and Second (= direct and converse parts of) Borel-Cantelli Lemmas.

   (b) Let $\{X_n\}$ be a sequence of independent and identically distributed random variables with zero mean and finite fourth moments. If $S_n = \sum_{k=1}^{n} X_k$, show that $\frac{S_n}{n} \to 0$, almost surely.