Preliminary Exam in Topology
January 2000

1. Let X be the quotient of the punctured Euclidean space $R^{n+1} - \{0\}$ by the equivalence relation $x \sim y$ if there exists $t > 0$ such that $y = tx$. Prove that X is homeomorphic to the unit sphere S^n in R^{n+1} with the subspace topology.

2. (a) State the least upper bound property of the real numbers.
(b) Use this property to prove that the unit interval $[0,1]$ is connected.

3. Let (X,d) be a complete metric space. Let $f : X \to X$ be a continuous map such that there is a real number $r \in (0,1)$ with

$$d(f(x), f(y)) \leq r\, d(x,y)$$

for all x, $y \in X$. Prove that f has a fixed point.

4. (a) Construct a 2-dimensional CW-complex X whose fundamental group has presentation $\langle a, b : a^2, b^4 \rangle$.
(b) Find all of the connected 2-sheeted covering spaces of X.
(c) Describe the universal covering space of X.

5. Prove that if X and Y are compact connected surfaces without boundary, with X nonorientable and Y orientable, then X is not a covering space of Y.

6. Classify compact connected surfaces S with Euler characteristic $\chi(S) \geq -2$. In other words, give a list of surfaces so that every compact connected surface S with $\chi(S) \geq -2$ is homeomorphic to a surface on this list, and no two surfaces on the list are homeomorphic. (The surfaces can be orientable or nonorientable, and they can have empty boundary or nonempty boundary.)

7. (a) What is the degree of the antipodal map $a_n : S^n \to S^n$ of the n-sphere? (You do not have to prove your answer.)
(b) Define a CW-complex X homeomorphic to real projective n-space RP^n.
(c) Using (a) and (b), compute the integral homology of RP^n. Show your work.

8. Let $X = CP^2$, the complex projective plane. Prove that if $f : X \to X$ is a continuous map homotopic to the identity map, then f has a fixed point.