Ph.D. Prelim: Probability Theory, January 2001

1. Show that the mean μ of a random variable X has the property

$$\min_c E(X-c)^2 = E(X-\mu)^2 = V(X).$$

2. (a) Prove that for any r.v. X

$$E|X| = \int_0^\infty P(|X| \geq t)dt.$$

   (b) Given a square integrable r.v. X, show that for $\lambda \geq 0$,

$$P(X - EX \geq \lambda) \leq \frac{\sigma^2(X)}{\sigma^2(X)+\lambda^2}.$$

3. Show that random variables X_n, $n \geq 1$, and X satisfy $X_n \to X$ in distribution iff

$$E[F(X_n)] \to E[F(X)]$$

   for every continuous distribution function F.

4. If $\{X_n\}$ are iid r.v.s, then $E|X_1| < \infty$ if and only if $\sum_{n=1}^\infty X_n \frac{\sin nt}{n}$ converges a.s. for every $t \in (-\infty, \infty)$.

5. Let $\{X_n\}$ be a sequence of independent random variables.

   (a) If $EX_n = 0$ for $n = 1, 2, \ldots$, and $\sum_{n=1}^\infty \mathrm{var}(X_n) < \infty$, show that $\sum_{n=1}^\infty X_n$ converges a.s.

   (b) State (without proof) Levy's inequality and use it to prove that $S_n = \sum_{k=1}^n X_k$ converges a.s. if and only if it converges in probability.