Algebra Prelim   January 2002

Do any of any six questions, including question I.

I. True or False? Tell whether each statement is true or false, giving in each case a brief of why, e.g. by a one or two line argument citing an appropriate theorem or principle, or counter example. Do not answer "this follows from X's theorem" without indicating why the hypotheses of X's theorem hold and what that theorem says in this case.

(i) A commutative ring R with identity $1 \neq 0$, always has a nontrivial maximal ideal M (i.e., such that $M \neq R$).

(ii) A group of order 100 has a unique subgroup of order 25.

(iii) A subgroup of a solvable group is solvable.

(iv) A square matrix over the rational numbers $\mathbb{Q}$ has a unique Jordan normal form.

(v) In a Noetherian domain, every non unit can be expressed as a finite product of irreducible elements.

(vi) If $F \subset K$ is a finite field extension, every automorphism of F extends to an automorphism of K.

(vii) A vector space V is always isomorphic to its dual space V^*.

(viii) If A is a real 3×3 matrix such that $AA^t = Id$, (where A^t is the transpose of A and Id is the identity matrix), then there exist mutually orthogonal, nonzero A-invariant subspaces V, W of $\mathbb{R}$.

In the following proofs give as much detail as time allows.

II. Do part (i) or (ii):

(i) If G is a finite group with subgroups H, K such that $G = HK$, and K is normal, prove G is the homomorphic image of a "semi-direct product" of H and K (and define that concept).

(ii) If G is a group of order pq, where $p < q$ are primes and p does not divide $q - 1$, prove G is isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_q$.

III. If F is a field, prove there is an extension field $F \subset K$ such that every irreducible polynomial over F has a root in K.

IV. Prove every ideal in the polynomial ring $\mathbb{Z}[x]$ is finitely generated where $\mathbb{Z}$ is the integers.

V. If n is a positive integer, prove the Galois group of $x^n - 1$ over the rational field $\mathbb{Q}$ is abelian.

VI. Do both parts (i) and (ii):

(i) State the structure theorem for finitely generated torsion modules over PID.

(ii) Set up (and explain) a one-to-one correspondence between the conjugacy classes of the elements of the group $GL(3, \mathbb{Z}_2)$ of invertible 3×3 matrices over $\mathbb{Z}_2$, and the following six sequences of polynomials: $(1+x, 1+x, 1+x), (1+x, 1+x^2), (1+x+x^2+x^3)$, $(1+x^3), (1+x+x^3), (1+x^2+x^3)$.

VII. Calculate a basis that puts the matrix $A = \begin{bmatrix} 8 & -4 \\ 9 & -4 \end{bmatrix}$ in Jordan form.

VIII. Given $k-$vector spaces A, B and $k-$linear maps $f : A \to A, g : B \to B$, with matrices $(x_{ij}), (y_{kl})$, in terms of bases $a_1, \ldots, a_n$, and $b_1, \ldots, b_m$, define the associated basis of $A \otimes B$ and compute the associated matrix $f \otimes g : A \otimes B \to A \otimes B$.