Numerical Analysis Preliminary Examination

Spring, 2002

Name____ Student Id. No._______________

Instruction: *Work any eight and only eight of the following ten problems. Please start each problem on a separate sheet of paper, write on only one side of the paper, and number each page. The time limit on this exam is three hours. Clearly strike out on this examination page the two problems that you wish to delete, and then hand in both this marked examination page and your solution pages for the eight problems that are to be graded.*

[1] Suppose $f(x) \in C[a,b]$ and that $f(a)f(b) < 0$. Derive an explicit formula for the number of bisections N, that are sufficient for the classical bisection method to yield a d-decimal place approximation to a root p of $f(x) = 0$.

[2] The error function **erf**(x), is defined through the integral

$$\mathbf{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt,$$

and can not be expressed in terms of elementary functions. Suppose $\mathbf{erf}(x)$ has been numerically tabulated for equally spaced values of x with step size $h = 0.1$ from $x = 0$ to $x = 1$. Use the classical interpolation error theorem to analytically determine the maximum error encountered if linear interpolation is used to approximate $\mathbf{erf}(x^*)$ for x^* a nontabular point in $[0, 1]$.

[3] Use Taylor's theorem for a function of two variables to carefully derive and state Newton's method for the numerical solution of the system of two nonlinear equations:

$$\begin{cases} f(x,y) = 0, \\ g(x,y) = 0. \end{cases}$$

[4] Let $S(x)$ be a spline of degree ≤ 1 that interpolates to the function $f(x)$ at the equally spaced knots:

$$x_i = a + ih, i = 0, \cdots, N,$$

where $h = (b-a)/N$. Derive an explicit formula for the definite integral $\int_a^b S(x)dx$ and simplify your answer as much as possible. Does the resulting quadrature formula correspond to any of the classical composite quadrature formulas? Explain.

[5] (a) Definite what is meant by the condition number $K_\infty(A)$ (relative to the ℓ_∞ subordinate matrix norm) of an $n \times n$ matrix A. (b) Compute $K_\infty(A)$ for the particular matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

[6] Consider the $n \times n$ linear system of equations $Ax = b$, where A is strictly diagonally dominant. There are many ways to write $Ax = b$ as an equivalent linear system of the form $x = Bx + c$. Clearly determine and state an appropriate choice for the iteration matrix B that will guarantee the convergence of iteration scheme

$$x_{n+1} = Bx_n + c, n = 0, 1, \cdots,$$

to the solution vector x for the system for any initial guess x_0. Explain your logic with care.

[7] Let $X = C[0,1]$ be normed by $\|\cdot\|_2$. Compute the best approximation to x^4 by a polynomial of degree ≤ 2.

[8] (a) Clearly explain exactly why the LU factorization of an $n \times n$ matrix A is useful for efficiently solving an $n \times n$ linear system $Ax = b$. (b) Use the LU factorization method to solve the following linear system

$$\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \\ 14 \end{bmatrix}.$$

[9] Determine the parameters a, b, α and β so that the one-step method defined by

$$y_{n+1} = y_n + ak_1 + bk_2$$

$$k_1 = hf(x_n, y_n),$$

$$k_2 = hf(x_n + \alpha h, y_n + \beta k_1)$$

has maximal local truncation error.

[10] Prove: if $f(x) \in C^2[a,b]$ then the remainder in the elementary trapezoidal rule

$$\int_a^b f(x)dx = \frac{b-a}{2}[f(a) + f(b)] + R(f)$$

can be expressed in the classical form

$$R(f) = -\frac{1}{12}(b-a)^3 f^{(2)}(\xi), \xi \in [a,b].$$