Topology Qualifying Exam
January 2003

1. A space X is *sequentially compact* if every sequence in X has a convergent subsequence. Prove that a compact metric space X is sequentially compact.

2. Prove that if X is Hausdorff and Y is a retract of X then Y is closed in X.

3. Describe the universal covering space of the Klein bottle and the group of deck transformations of this covering space.

4. Find all the connected 2-sheeted covering spaces of the figure 8 (the wedge product of two circles). Use this classification to find all the index 2 subgroups of the free group on two generators.

5. Find all the compact orientable surfaces which are covering spaces of a compact orientable surface of genus 3. (Hint: If X is an n-sheeted covering space of the finite CW complex Y, then the Euler characteristic of X is n times the Euler characteristic of Y.)

6. Use the Mayer-Vietoris sequence to compute the homology of the Klein bottle by writing the Klein bottle as the union of two Möbius strips along their boundaries.

7. Prove that for all $n > 0$ the unit sphere S^{n-1} in Euclidean n-space $\mathbb{R}^n$ is not a retract of $\mathbb{R}^n$.

8. State the Lefschetz Fixed Point Theorem. Prove that if f is a continuous map from complex projective n-space to itself, $n \geq 0$, and f is homotopic to the identity map, then f has a fixed point. If you know the homology groups of $\mathbb{C}P^n$, you do not need to rederive them.