Algebra Prelim

Spring 2004

1. Classify all groups of order 18, up to isomorphism.

2. Use the class equation to prove that every nontrivial group of prime power order has a nontrivial center.

3. Let R be a commutative ring, $I \subset R$, and R_I the localization of R at I, i.e., the ring of fractions obtained by inverting all elements of R not in I. Prove that the ideal

$$\{\frac{a}{b} \,|\, a \in I, b \notin I\} \subset R_I$$

is maximal, and that it is the unique maximal ideal.

4. Prove that

   (a) the ring $\mathbb{Z}[i]/(2)$ is not a field,

   (b) the ring $\mathbb{Z}[i]/(3)$ is a field.

5. (a) Let R be a commutative ring, and let

$$0 \to A \to B \to C \to 0$$

   be a short exact sequence of R-modules. Prove that if A and C are finitely generated, then B is finitely generated.

   (b) Give an example of a commutative ring R and a short exact sequence of R-modules

$$0 \to A \to B \to C \to 0$$

   such that B and C are finitely generated and A is not finitely generated.

6. Let A be a nilpotent matrix with real entries (i.e., $A^n = 0$ for some $n > 0$). Prove that A has trace 0.

7. Let $GL^+(2, \mathbb{R})$ denote the set of 2×2 matrices with real entries and positive determinant. Show that for every matrix $A \in GL^+(2, \mathbb{R})$ there exists a matrix $B \in GL^+(2, \mathbb{R})$ such that $B^2 = A$.

8. In each of the following cases, determine whether the given field extension is Galois, determine the degree of the extension if it is Galois, and if it is Galois, describe the Galois group.

   (a) The splitting field of the polynomial $x^3 - 2$ over $\mathbb{Q}$.

   (b) The extension of $\mathbb{Q}$ formed by adjoining the real cube root of 2.

   (c) The splitting field over $\mathbb{F}_p(t)$ of the polynomial $f(x) = x^p - t$, where p is prime and $\mathbb{F}_p(t)$ is the field of rational functions over $\mathbb{F}_p$.

9. Find the minimal polynomial of the matrix

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$