Topology Qualifying Examination
August, 2015

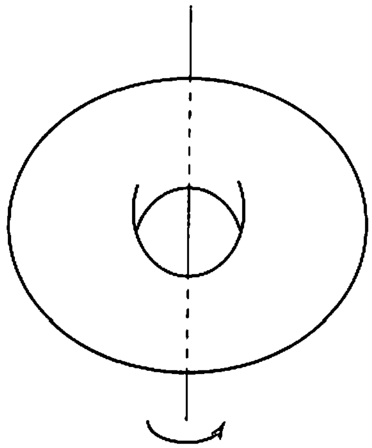
Instructions: Work all problems. Give clear explanations and complete proofs.

(1) Show that if $A$ and $B$ are compact, then so is $A \times B$.

(2) Given two points $a$ and $b$ of a space $S$, a collection of sets $A_1, \ldots, A_n$ in $S$ is called a *simple chain from $a$ to $b$* if $a \in A_1$, $b \in A_n$, and $A_i \cap A_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Prove that if $\{U_\alpha\}$ is a collection of open sets covering $S$ and $S$ is a connected space, then there is a simple chain of elements in $\{U_\alpha\}$ joining $a$ and $b$ for any $a$ and $b$. Hint: Consider the set $C_a$ of all $b$ so that there is a simple chain of elements joining $a$ and $b$. Prove that $C_a$ is open and closed.

(3) Let $\tau$ be the map that in cylindrical coordinates takes $(r, \theta, z)$ to $(r, \theta + \pi, z)$, thus $\tau$ maps the torus $T$ to itself as shown below.



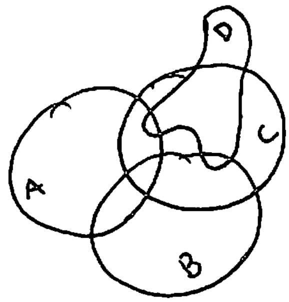
The map $\tau$ from the torus to itself.

a) Find a cell structure on $T$ such that $\tau$ maps cells to cells.

b) Let $Q$ be the quotient of $T$ given by identifying $x$ and $\tau(x)$ for all $x \in T$. What is $\chi(Q)$?

c) Is $Q$ a surface?

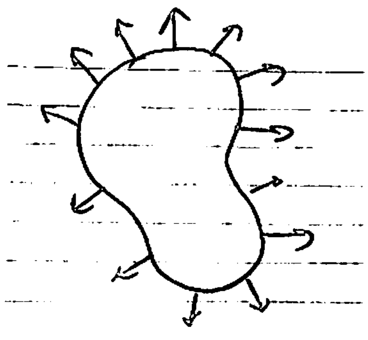
(4) The *nerve* of a collection of sets is an abstract simplicial complex that has a 0-cell for each set, a 1-cell joining each pair of sets that intersect each other, a 2-cell for every triple of sets with a common intersection, and so forth. Consider the collection of sets below:




Now

(a) Draw the nerve of the collection.
(b) Compute the homology of the nerve.

(5) A continuous vector field $V$ on the plane is a continuous map from $\mathbb{R}^2$ to $\mathbb{R}^2$. The portion of $V$ along the boundary of a disk $D$ is shown below:



Show that the vector field has a zero inside $D$.

(6) Express a Klein bottle as the union of two annuli. Use the Mayer-Vietoris squence and this decomposition to compute its homology.

(7) Compute the fundamental group, using any technique you like, of $\mathbb{R}P^2 \# RP^2 \# RP^2$.

(8) Explicitly give a collection of deck transformations on $\{(x, y) | -1 \leq x \leq 1, -\infty < y < \infty\}$ such that the quotient is a Mobiüs band.