Topology Qualifying Exam Fall 2007

Instructions: Work all problems. Give clear explanations and complete proofs.

1. Let Y be a compact subset of a Hausdorff space X. Prove that $X \setminus Y$ is open (and hence that Y is closed).

2. Let $\{X_\alpha : \alpha \in \mathcal{A}\}$ be a family of connected subspaces of a space X such that there is a point $p \in X$ which is in each of the X_α. Show that the union of the X_α is connected.

3. Find all the connected 2-sheeted covering spaces of the wedge product of two circles $S^1 \vee S^1$. Also, find all the index 2 subgroups of the free group on two generators, and explain the relationship between these problems.

4. Prove that every continuous map $f : \mathbb{R}P^2 \to S^1 \times S^1$ is null-homotopic.

5. (a) State van Kampen's Theorem.

(b) Calculate the fundamental group of the space obtained by taking two copies of the torus $T = S^1 \times S^1$ and gluing them together along a circle $S^1 \times \{p\}$, where p is a point in S^1.

6. Describe a cell complex structure on the torus $T = S^1 \times S^1$ and use this to compute the homology groups of T. (To justify your answer you will need to consider the attaching maps in detail.)

7. Let X be the space obtained as the quotient of a disjoint union of a 2-sphere S^2 and a torus $T = S^1 \times S^1$ by identifying the equator in S^2 with a circle $S^1 \times \{p\}$ in T. Compute the homology groups of X.

8. Let X be a compact orientable surface (without boundary) and let $f : X \to X$ be a continuous map. Assume that f is homotopic to the identity map but that f has no fixed points. What is the genus of X? Prove your answer.