UGA TOPOLOGY QUALIFYING EXAM
FALL 2011

1. Let X be a topological space, and $B \subset A \subset X$. Equip A with the subspace topology, and write $cl_X(B)$ or $cl_A(B)$ for the closure of B as a subset of, respectively, A or X. Determine, with proof, the general relationship between $cl_A(B)$ and $cl_X(B) \cap A$ (*i.e.*, are they always equal? is one always contained in the other but not conversely? neither?)

2. Give examples of:
 (a) A connected space which is not path-connected
 (b) A path-connected space which is not locally connected.

3. (a) State what it means for a topological space X to be
 (i) compact
 (ii) Hausdorff
 (iii) normal
(b) Prove that every compact Hausdorff space is normal.

4. Let $V = D^2 \times S^1 = \{(z, e^{it}) | z \in \mathbb{C}, |z| \leq 1, 0 \leq t < 2\pi\}$ be the "solid torus," with boundary given by the torus $T = S^1 \times S^1$. For $n \in \mathbb{Z}$ define $\phi_n \colon T \to T$ by $\phi_n(e^{is}, e^{it}) = (e^{is}, e^{i(ns+t)})$. Find the fundamental group of the identification space

$$X_n = \frac{V \coprod V}{\sim_n}$$

where the equivalence relation $\sim_n$ identifies a point x on the boundary T of the first copy of V with the point $\phi_n(x)$ on the boundary of the second copy of V.

5. Prove that, for $n \geq 2$, every continuous map $f \colon \mathbb{R}P^n \to S^1$ is null-homotopic.

6. Exhibit a cell decomposition of the Klein bottle and use this to compute its homology.

7. For any $n \geq 1$, let $S^n = \left\{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \,\middle|\, \sum x_i^2 = 1\right\}$ denote the n-dimensional unit sphere and let $E = \{(x_0, \ldots, x_n) \in S^n | x_n = 0\}$ denote the "equator." Find, for all k, the relative homology $H_k(S^n, E)$.

8. For any natural number g let Σ_g denote the (compact, orientable) surface of genus g. Determine, with proof, all numbers g with the property that there exists a covering space $\pi \colon \Sigma_5 \to \Sigma_g$. (Hint: How does the Euler characteristic behave for covering spaces?)