Fall 2016 Topology Qualifying Exam

Work all problems, justify your calculations, and explicitly state which theorems you are using. Each problem is worth 10 points.

1. Let $\mathcal{S}, \mathcal{T}$ be topologies on a set X. Show that $\mathcal{S} \cap \mathcal{T}$ is a topology on X. Give an example to show that $\mathcal{S} \cup \mathcal{T}$ need not be a topology.

2. Prove that a metric space X is normal, i.e. if $A, B \subset X$ are closed and disjoint then there exist open sets $A \subset U \subset X, B \subset V \subset X$, such that $U \cap V = \emptyset$.

3. Let S_k be the space obtained by removing k disjoint open discs from the sphere S^2, to leave a surface whose boundary is k circles. Form X_k by gluing k Möbius bands onto S_k , one for each circle boundary component of S_k (by identifying the boundary circle of a Möbius band homeomorphically with a given boundary component circle). Use Van Kampen's theorem to calculate $\pi_1(X_k)$ for each $k > 0$ and identify X_k in terms of the classification of surfaces.

4. Show that if $p : X \to \mathbb{C}P^n$ is a covering space map, then X is homeomorphic to $\mathbb{C}P^n$.

5. Let A be the union of the unit sphere in $\mathbb{R}^3$ and the interval $\{(t, 0, 0) : -1 \leq t \leq 1\} \subset \mathbb{R}^3$. Compute $\pi_1(A)$ and give an explicit description of the universal cover of A.

6. Let Σ be a closed orientable surface of genus g. Compute $H_i(S^1 \times \Sigma; \mathbb{Z})$ for $i = 0, 1, 2, 3$.

7. Let X be the topological space obtained as the quotient of the sphere $S^2 = \{\mathbf{x} \in \mathbb{R}^\mathbf{3}, \|\mathbf{x}\| = \mathbf{1}\}$ under the equivalence relation $\mathbf{x} \sim -\mathbf{x}$ for $\mathbf{x}$ in the equatorial circle, i.e. for $\mathbf{x} = (x_1, x_2, 0)$. Calculate $H_*(X, \mathbb{Z})$ from a CW complex description of X.

8. Prove the Brouwer fixed point theorem. In other words, prove that every continuous map from D^2 to itself has a fixed point (without using either the Brouwer or Lefschetz fixed point theorems).