Goals of 8000 course: To prepare students to use the basic tools of commutative and non commutative algebra, and pass the PhD alg. prelim.

Proposed content of course:
We will treat commutative topics first, generalizing vector spaces. The fundamental concept is "linear combinations".

I. Linear and commutative algebra

i) Abelian groups

First we treat abelian groups, representing them as cokernels of maps between free abelian groups, especially finitely generated ones using matrices. Modifying familiar techniques for matrices, elementary row and column operations, let us diagonalize integer matrices and prove the fundamental structure theorem, giving explicit models of all finitely generated abelian groups as products of cyclic groups.
Key properties peculiar to abelian groups include the ability to form quotients by modding out any subgroup, the existence of an element whose "order" is the l.c.m. of the orders of any two given elements, and especially the ability to define maps out of free abelian groups by defining them arbitrarily on the basis elements. In particular a finite abelian group has elements of every order dividing the annihilator of the group, and subgroups of every order dividing the order of the group.

ii) Commutative rings R

Then we discuss rings more general than the integers, making each important theorem about integers into a definition.
We call a ring "entire" or "integral domain" or "domain", if there are no zero divisors, a Euclidean domain if it admits a Euclidean algorithm (e.g. polynomials in one variable over a field, Gaussian integers), a "p.i.d." if also every "ideal" (subgroup closed under multiplication by R) has a single generator (e.g. local rings of smooth curves), and a u.f.d. if it also admits unique factorization into irreducibles (e.g. polynomials in any number of variables over a field or over any u.f.d.).
R is "noetherian" if every ideal has a finite set of generators (e.g. any quotient ring of any polynomial ring over a field or any noetherian ring). In a noetherian domain factorization into irreducibles is always possible, but maybe not uniquely - uniqueness is equivalent to the existence of gcd's for any pair of elements.
The rational root theorem leads to calling a domain R "integrally closed" or "normal" if every root of a monic polynomial which lies in its fraction field, already lies in R (e.g. the coordinate ring of an affine hypersurface which is smooth in codimension one).
A "dedekind domain" is a normal domain where (as in Z) every proper prime ideal is maximal, (e.g. the affine ring of a smooth curve, or the ring of integers in a number field; algebraic number theory is more difficult apparently because these rings do not always have the stronger properties listed previously).

iii) R modules

Since our study of abelian groups used crucially the multiplication of group elements by integers, analogous to multiplying vectors by scalars, we define "R modules" as abelian groups which allow multiplication by elements of a ring R. For rings R which share those properties of Z used in the proof of the fundamental structure theorem, we obtain analogous theorems for R modules.

Every finitely generated module over a noetherian ring is the cokernel of a matrix. Every matrix over a Euclidean domain can be diagonalized by elementary row and column operations, and invertible secondary operations suffice to diagonalize matrices over a p.i.d. Hence we get structure theorems for finitely generated modules over all p.i.d.'s, and an algorithm for computing the decomposition over a Euclidean domain.
Presumably there are some theorems for modules over normal and dedekind domains, but I do not know what they are.

iv) Canonical forms of linear operators

Applications of these theorems include the important case of finitely generated "torsion" modules (analogous to finite groups) over k[X] where k is a field, since these are equivalent to pairs (V,T) where V is a finite dimensional k - vector space, and T is a k linear transformation. The structure theorem above gives as a result, the rational and Jordan canonical forms for T, (with certain hypotheses), as well as diagonalization criteria for the matrix of T.
We point out analogies between the order of a finite group and the characteristic polynomial of T, and between the annihilator of a finite group and the minimal polynomial of T. The Cayley Hamilton theorem falls out too.
This completes the first half of the course.

II. Non commutative algebra: groups and field extensions

The basic concept in non abelian group theory is "conjugation", studying the extent to which the action sending y to a^-1(y)a is non trivial.

i) Groups, Existence of subgroups, (Sub)normal towers

The first goal is to understand something about the elements and subgroups of a given group, just from knowing its order.
We begin the study of possibly non abelian groups, motivated by a desire to understand not just individual matrices, but groups of matrices, as well as symmetries of both geometric and algebraic objects, e.g. field extensions. Fresh difficulties here include the fact that not all subgroups can be modded out to form quotient groups, i.e. not all subgroups are "normal", and the related problem that not all subgroups can be kernels of homomorphisms, so it is harder to find non trivial maps between groups, hence harder to compare groups.
Even non normal subgroups are harder to find, as it is no longer true that there are subgroups of all orders dividing the order of a finite G, nor elements whose order equals the lcm of the orders of two given ones. Hence an initial problem is finding the orders of elements in a given G and the orders of subgroups. We recover some general results by restricting to prime power divisors. I.e. Sylow: for every prime power p^r dividing #G, there exist subgroups of G of order p^r and elements of order p.
As a substitute for the product decomposition of a finite abelian group into cyclic groups, we have the concept of a "simple" group (only trivial normal subgroups), and of a "subnormal" tower for G in terms of simple constituents which are uniquely determined by G.

ii) Free groups, Group actions

After getting some handle on the elements and subgroups of a given group, we seek to construct homomorphisms of groups.
Defining a homomorphism out of G is most naturally accomplished by finding an "action" of G

on a set S, which yields a homomorphism of G into Sym(S) the group of bijections S--->S or "symmetries" of S.

A non abelian group is characterized by the fact that some conjugation actions are non trivial. Letting G act on its Sylow subgroups by conjugation or translation, can provide non trivial homomorphisms and non trivial normal subgroups. These actions are also used in the proof of the Sylow theorems.

From this perspective, the structure theorems for linear transformations give unique standard representatives useful for computing the orbits of conjugation actions on GLn(k). This lets us understand GL3(Z/2) = collineations of the 7 point plane, the next interesting simple group after the icosahedral rotation group A(5).

Free abelian groups, from which homomorphisms to abelian groups are easy to define, must be replaced by "free [non abelian] groups", which allow easy homomorphisms to all groups, but whose structure is much harder to understand. Thus although every finite group G is the quotient of a free group by a (free) subgroup, this information is harder to use as the presentation of G by a map between free groups is harder to simplify, since matrices are inapplicable. M. Artin's discussion of the Todd - Coxeter algorithm is presumably relevant here. We show finitely generated free groups are fundamental groups of certain covering spaces, and use this to prove all subgroups of a free group are free.

iii) Semi direct products

To classify even small non abelian groups, we need some standard examples and some standard constructions. Basic examples include symmetric groups and dihedral groups. Most non abelian groups do not decompose as direct products, but we learn to recognize those which do. We introduce a more general "semi direct" product, and show that many small groups do decompose as "semi direct" product of still smaller groups, and learn to recognize when a group has such a decomposition. Semi direct products of abelian groups can be non abelian. Dihedral groups D(2n) are semi direct products of two cyclic groups, Z/2 and Z/n.

iv) Application to Galois groups of field extensions

Galois honed the tool of group theory to analyze the structure of finite field extensions, to decide when a polynomial with roots in an extension of k, could be expressed in terms of various "nth roots" and field operations, i.e. whether the a sequence of linear operations and forming powers, can be inverted by a sequence of rational operations and taking roots.

The answer is given by analyzing a subnormal tower of a subgroup G of all field automorphisms of the full root field, namely those which leave the coefficient field point wise fixed. Solvability of the polynomial is equivalent (over Q say) to a subnormal tower for G having only abelian simple constituents.

As a technically convenient device, we will construct a universal algebraic extension of a given base field, its "algebraic closure", using Zorn's lemma. Then we can always work within this one field and with quotients of its Galois group. We also introduce the concept of transcendence degree, as an invariant of non algebraic extensions, and prove it is well defined (at least when finite) by the classical "replacement lemma".

v) Examples, computations of Galois groups

We practice computing a few small Galois groups, including some of form D(2n), S(n), and we recall the structure of a subnormal tower for these groups, relating it to solvability of polynomials. In particular we deduce Abel’s theorem that a general polynomial of degree 5 or more is not solvable by radicals. Then we discuss some fields associated to the cyclotomic polynomials X^n - 1, whose Galois groups are abelian. We define the discriminant, show how it reveals whether Gal is contained in A(n), and compute it for normalized cubics.