845 course notes part 3,

§13) Hom, Duality, and Representable Functors

In many areas of mathematics we study objects by means of their "mapping properties", i.e. by how they map into other objects, or how other objects map into them. For example, we count finite sets by mapping them injectively into the natural numbers, classify compact surfaces by mapping polygons onto them, "represent" groups by mapping them into permutation or matrix groups, classify covering spaces by mapping "loops" into them, study field extensions in Galois theory by mapping them into themselves, and classify vector bundles on a space by mapping that space into a "Grassmanian". In addition to Galois groups, the fundamental group and the singular homology and cohomology groups are either subgroups, quotient groups or "subquotients" of collections of maps. Thus it will be useful for us to familiarize ourselves with the general properties of sets of maps, and with the fundamental operation on maps, composition. We will focus on R-module maps, but some of our results are valid for other categories of maps. Given R modules M,N we will study not only the module $Hom_R(M,N)$, but the functors $Hom_R(M,\cdot)$ and $Hom_R(\cdot,N)$, and their action on homomorphisms. The examples above may help convince you that of all the functors in the world, Hom is perhaps the most important. [This is a good time to go back and reread section 843, I, §10, on categories and functors.]

Recall that the functor $Hom_R(M,\cdot)$, which we will usually write as $Hom(M,\cdot)$, assigns to a module X the module Hom(M,X), and to a map $f:X\to Y$ the map $f_*:Hom(M,X)\to Hom(M,Y)$ which takes g to $f_*(g) = f\circ g$. Thus f_* means "follow by f". Dually, the functor $Hom(\cdot,N)$ takes X to Hom(X,N), and $f:X\to Y$ to $f^*:Hom(Y,N)\to Hom(X,N)$, where $f^*(g) = g\circ f$, i.e. f^* means "precede by f". In general, if F is a functor such that $f:X\to Y$ induces $F(f): F(X)\to F(Y)$ (direction preserving), we call F "covariant", while if $f:X\to Y$ induces $F(f): F(Y)\to F(X)$ (direction reversing), we call F "contravariant". Thus $Hom(M,\cdot)$ and "lower star" is a covariant functor, while $Hom(\cdot,N)$ and "upper star" is contravariant. In particular, $(f\circ g)_* = f_*\circ g_*$, while $(f\circ g)^* = g^*\circ f^*$. [Confusingly, in topology "homology" is covariant and "cohomology" is contravariant. Once Hilton and Wylie quite plausibly suggested changing the latter term to "contrahomology", but as in most well meaning attempts to change tradition, this failed totally. In differential geometry also, I believe "contravariant tensors" are actually covariant. So be vigilant as always. Maybe we could start a movement to call them "sense preserving" or "sense reversing"!

Notation: The category of R-modules and homomorphisms is denoted by $\mathfrak{M}$, or $\mathfrak{M}_R$, the category of sets and functions by $\mathcal{S}$, and a functor F from modules to sets, or modules to modules, may be denoted by $F:\mathfrak{M}\to\mathcal{S}$ or $F:\mathfrak{M}\to\mathfrak{M}$.

Remark: The functors Hom(X,·) and Hom(·,X) may be considered as module valued functors $\mathfrak{M}\to\mathfrak{M}$ or as set valued functors $\mathfrak{M}\to\mathcal{S}$.

Dual Modules and abstract adjoints

The functor Hom(·,R) which assigns to M the "dual module" $M^* =$ Hom(M,R), is called the dual functor. To a map $T:M\to N$ it associates the map $T^*:N^*\to M^*$, "preceding by T". There is a good reason this looks a lot like the notation used in the study of hermitian and inner product spaces. Recall that a complex hermitian space $(N,\langle,\rangle)$ comes equipped with a conjugate linear isomorphism $N\to N^*$ taking w to $\langle\cdot,w\rangle:N\to\mathbb{C}$. Preceding this map by T yields the element $\langle T(\cdot),w\rangle:M\to\mathbb{C}$. As before there is a unique element $T^*(w)$ of M such that $\langle T(\cdot),w\rangle = \langle\cdot,T^*(w)\rangle$ as functions on M. Thus the previously considered hermitian adjoint map $T^*:N\to M$ is identified with the new composition map $T^*:N^*\to M^*$ via the (conjugate linear) isomorphisms $M\cong M^*$ and $N\cong N^*$ coming from the hermitian products in M and N.

The analogous assertions hold in a real inner product space. In $\mathbb{R}^n$ where we have a distinguished basis as well as an inner product, the abstract adjoint $T^*:(\mathbb{R}^n)^*\to(\mathbb{R}^m)^*$ of a map $T:\mathbb{R}^m\to\mathbb{R}^n$, is identified under the usual isomorphism $(\mathbb{R}^n)^* = \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}) \approx \mathbb{R}^n$ [which is the same as the isomorphism induced by the inner product], with the map $T^*:\mathbb{R}^n\to\mathbb{R}^m$ defined by the transpose of the matrix of T. In particular, $[T^*] = [T]^t$ and $[ST]^t = [(ST)^*] = [T^*S^*] = [T^*][S^*] = [T]^t[S]^t$. (Notice this is an easy way to check that the transpose of a product equals the product of the transposes, but in the opposite order.)

In general, without any dot products, and over any ring, the functor $Hom(\cdot,R)$ provides an abstract "adjoint" of any map $T:M\to N$ of any (not necessarily free) modules, namely the adjoint is the map $T^*:N^*\to M^*$ where $T^*(f) = f\circ T$. We can also define the "orthogonal complement" of a submodule $N\subset M$, to be the submodule $N^\perp\subset M^*$ where f is in $N^\perp$ iff $N\subset ker(f)$. I.e. a function $f:M\to R$ is "orthogonal" to N iff it vanishes on N. Our earlier principle that the orthogonal complement of a T-invariant subspace is T^*-invariant has an analog here too. Namely if $T:M\to Y$, and if for some submodule $N\subset M$ we have $T(N)\subset X\subset Y$, then $T^*(X^\perp)\subset N^\perp\subset M^*$. In particular, for an endomorphism $T:M\to M$, if $T(N)\subset N\subset M$, then $T^*(N^\perp)\subset N^\perp\subset M^*$

Remember: Whenever the vector spaces under discussion are not inner product spaces, then T^*, and $N^\perp$ must be given the abstract meanings described here.

Dual Bases (in finite dimensions)

We have already observed that for every ring R, the functor $Hom(R,\cdot)$ is equivalent to the identity functor, in particular $Hom(R,N) \cong N$ for every R-module N. The situation is quite different for the dual functor $Hom(\cdot,R)$. The example $Hom_{\mathbb{Z}}((\mathbb{Z}_{19})^{100}, \mathbb{Z}) = \{0\}$, shows that the "dual module" N^* may contain very little information about N. It is most informative for finite free modules $M \cong R^n$. In that case, we have $M^* = Hom(M,R) \cong Hom(R^n,R)\cong R^n \cong M$, a special case of the isomorphism $Hom_R(R^n,X) \cong X^n$, for every X. These isomorphisms depend on a choice of basis of M. The point is that an isomorphism $M \cong R^n$ is equivalent to a basis for M, and that this basis determines a dual basis for M^*, which in turn determines an isomorphism between M^* and R^n. Composing the two isomorphisms of $M \cong R^n \cong M^*$ gives an isomorphism $M \cong M^*$. Different bases give different isomorphisms in general.
Let's give these isomorphisms explicitly. Recall that the usual isomorphism $Hom(R^n,R) \cong R^n$ takes f to $(f(e_1),\ldots,f(e_n)) = \Sigma f(e_j)e_j$. Thus if $\{v_1,\ldots,v_n\}$ is a basis for M, the associated isomorphism $M^* = Hom(M,R) \cong M$ takes f to $\Sigma f(v_j)v_j$. This means that f_j corresponds to v_j iff $f_j(v_j) = 1$, and $f_j(v_i) = 0$ for $i\neq j$. The subset $\{f_1,\ldots,f_n\}\subset M^*$ defined by these properties is called the basis "dual" to the basis

$\{v_1,\ldots,v_n\}$ for M. That $\{f_1,\ldots,f_n\}$ is a basis of M^* follows from the fact that the isomorphism above $M \to M^*$ takes v_j to f_j, and $\{v_j\}$ is a basis of M. Each of these bases is useful for expressing elements of the *other* space in terms of the *other* basis. I.e. if v is in M, then $v = \Sigma f_j(v)v_j$, and if f is in M^* then $f = \Sigma f(v_j)f_j$

To see that the isomorphism $M \to M^*$ above depends on the choice of basis of M, just change the basis by replacing v_1 by $\tilde{v}_1 = 2v_1$. Then $\tilde{v}_1$ corresponds to the function $\tilde{f}_1$ which has value 1 on $\tilde{v}_1 = 2v_1$. Hence $v_1 = (1/2)\tilde{v}_1$ now corresponds to $(1/2)\tilde{f}_1$ which has the value 1/4 on v_1. Thus v_1 corresponds to a different function under the new isomorphism.

Double duality

Interestingly, for any module M there is a natural map $M \to M^{**}$, the "evaluation map" taking v to e_v = "evaluation at v". I.e. for each v in M, e_v is the element of $M^{**} = \mathrm{Hom}(M^*,R)$ that takes f in M^* to f(v) in R. If M is free of finite rank, then we know $M \cong M^* \cong M^{**}$, and our question is whether the *natural map* $v \mapsto e_v$ is an isomorphism. To see it is injective, we must show that for $v \neq 0$, there is some f in M^* such that $f(v) \neq 0$. We may use a basis $\{v_1,\ldots,v_n\}$ of M to construct such an f. Namely, $v = \Sigma a_j v_j$ where some $a_j \neq 0$, and then the function f_j in the dual basis defined above does the trick, since $f_j(v) = a_j$. To see ontoness, let $\lambda: X^* \to R$ be any map, and again use a basis to find a v such that $\lambda = e_v$. Namely, let $\{v_1,\ldots,v_n\}$ be a basis for M, let $\{f_1,\ldots,f_n\}$ be the dual basis, and put $v = \Sigma\lambda(f_j)v_j$. Then for any f in M^*, $f = \Sigma f(v_j)f_j$, so $\lambda(f) = \lambda(\Sigma f(v_j)f_j) = \Sigma f(v_j)\lambda(f_j) = \Sigma\lambda(f_j)f(v_j) = f(\Sigma\lambda(f_j)v_j) = f(v) = e_v(f)$. It follows that the natural map $M \to M^{**}$ is indeed an isomorphism when M is a finite free R module.

Exercise #162) a) Prove if N is a "torsion" module [i.e. if for every x in N, there is an $r \neq 0$ in R with $rx=0$], and R a domain, then $N^*=0$.
b) Describe N^* for any finitely generated **Z**-module N.

Exercise #163) If $\{x_j\}j=1,\ldots,m$ is a finite basis for a vector space M, prove directly from the definition (of basis) that the set $\{f_j\}j=1,\ldots,m$ defined by $f_j(v_j) = 1$, and $f_j(v_i) = 0$ for $i \neq j$, is a basis of M^*.

Exercise #164) If X,Y,Z are finite dimensional vector spaces, prove:
a) If $T:X\to Y$ is linear, with adjoint $T^*:Y^*\to X^*$, then $\ker(T^*) = (\mathrm{im}(T))^\perp \subset Y^*$.
b) If $Z\subset X$ is a subspace, and $Z^\perp\subset X^*$, then $\dim(Z)+\dim(Z^\perp) = \dim(X)$.

Exercise #165) For any finite dimensional vector spaces X,Y and any linear map $T:X\to Y$, the natural isomorphisms $X\cong X^{**}$ and $Y\cong Y^{**}$ identify the map T with its double adjoint $T^{**}:X^{**}\to Y^{**}$. I.e. if $\Theta_X:X\to X^{**}$ and $\Theta_Y:Y\to Y^{**}$ are the natural "evaluation" maps [taking x to e_x and y to e_y], then the following compositions are equal: $\Theta_Y\circ T = T^{**}\circ\Theta_X$ [I.e. the double dual functor is equivalent to the identity functor, on finite dimensional vector spaces. The same is true for finite free R-modules.]

The dual of an infinite dimensional vector space
The discussion above shows that if k is a field, and N is a finite dimensional k-vector space, then the dual space $N^* = \mathrm{Hom}_k(N,k)$ is always isomorphic to N, and any choice of basis of N determines such an isomorphism. If N is an *infinite* dimensional vector space, this is no longer true. Indeed suppose $\{v_j\}_{j=1,\ldots,\infty}$ is a countable vector basis for N. Then each element v of N is uniquely expressible as a finite linear combination $v = \Sigma a_jv_j$ of these v_j. Letting v correspond to the sequence of coefficients in this expression gives a one-one correspondence between the elements of N and those infinite sequences $\{a_j\}$ of elements of k in which all but a finite number of the entries a_j are equal to zero. Equivalently, N is in one-one correspondence with the set of all finite sequences of elements of k, where the last entry in a sequence must be nonzero. In particular, if k is a countable or finite field, the set of all sequences of length n is countable for every n, so the set of all such finite sequences is countable, and hence N is countable. On the other hand, we claim $N^* = \mathrm{Hom}(N,k)$ is not countable even if $k = \mathbb{Z}_2$. To see that, recall that a linear map $f:N\to k$ is determined by its values on a basis, and that we may define a map which sends the basis anywhere we please. Thus f is determined by the infinite sequence of its values $\{f(v_1), f(v_2),\ldots\ldots\}$, and that sequence can be any sequence of elements of k, with no restriction that most of them be

zero. Hence N^* is in one-one correspondence with the set of all infinite sequences of elements of k, which is an uncountable set even for k = $\mathbf{Z}_2$. [Recall the argument: if $\lambda:\mathbf{Z}^+ \to N^*$ is any function, define a sequence $\{a_j\}$ where $a_j = 0$ if $(\lambda(j))(v_j) = 1$, and $a_j = 1$ if $(\lambda(j))(v_j) = 0$. Then the function f in N^* defined by $f(v_j) = a_j$ for all j, cannot be in the image of λ, since for every j, $\lambda(j)$ and f have different values on v_j. Thus no function $\lambda:\mathbf{Z}^+ \to N^*$ can be surjective, so N^* is uncountable.] In particular the natural map $M \to M^{**}$ is not always an isomorphism, even for vector spaces.

Remark: If S is any infinite set and M is a vector space over $\mathbf{Z}_2$ with S as basis, then M has the same cardinality as S, while (as pointed out in class yesterday by Patricio) M^* has the same cardinality as 2^S, the set of subsets of S, which is greater than the cardinality of S. Thus M is never isomorphic to M^*, nor is M even bijectively equivalent to M^*, if M is infinite dimensional over $\mathbf{Z}_2$.

Question: If M is a countably infinite dimensional vector space over $\mathbf{Z}_2$, with basis $\{x_j\}_{j=1,\ldots,\infty}$, then since M^* is a $\mathbf{Z}_2$ vector space it must be a free $\mathbf{Z}_2$-module, hence must have a vector basis. What is that basis? Note: the basis must be uncountable. If we just take the "dual basis" of the basis $\{x_j\}$ we get a countable set of functions $\{f_j\}$ such that $f_j(x_j) = 1$, and $f_j(x_i) = 0$ when $i \neq j$. Thus this cannot be a vector basis of M^*.

[Note that in some sense however, every element of M^* is an *infinite* "linear combination" of the functions $\{f_j\}$. I.e. if f is a function with values $f(x_j) = a_j$, then in some sense f = "$\Sigma_{j=1\ldots\infty}\, a_j f_j$". I.e. if $x = \Sigma b_i x_i$ (finite sum), we have $f(x) = \Sigma\, b_i a_i$, and on the other hand, if we agree that the sum of an infinite number of zeroes is zero, then $(\Sigma_{j=1\ldots\infty}\, a_j f_j)(x_i) = a_i$, so we also have $(\Sigma_{j=1\ldots\infty}\, a_j f_j)(x) = (\Sigma_{j=1\ldots\infty}\, a_j f_j)(\Sigma b_i x_i) = \Sigma\, a_i b_i = f(x)$!!]

Infinite sums versus infinite products

So what went wrong, in the infinite case, with the argument that was used in the finite free module case to show $M^* \cong M$? The argument already fails in the very familiar case $\mathrm{Hom}(R^n,R) \cong R^n$. For a free R-module of *infinite* rank, this is not necessarily true! I.e if M is an R-module which is free on a countable set S, then M may not be isomorphic to $M^* = \mathrm{Hom}(M,R)$. We could say $\mathrm{Hom}(R^\infty,R)$ is not isomorphic to R^∞, but we have to say what we *mean* by the notation R^∞. If we want R^∞ to denote a free R-module, *free on a countably* infinite set, it turns out we must define R^∞ to be the set of those sequences of elements of R in which all but a finite number of the elements of each sequence are zero. On the other hand, for the same reason as for vector spaces, then $\mathrm{Hom}(R^\infty,R)$ turns out to be the set of all arbitrary sequences of elements of R. The difference is related to one we have alluded to before, the distinction between infinite products and infinite sums, and this is as good a time as any to explain it.

The point is that we want a "sum" of several objects to be an object such that a map *out* of it corresponds precisely to one map out of each of the individual objects. Dually we want a "product" of several objects to be an object such that a map *into* it is determined by precisely one map into each of the individual objects. As we know, a finite product of modules has both of these properties, but no one object has both these properties with respect to an infinite collection of modules. Let's be more precise. Recall that the **cartesian product** $\prod_A S_\alpha$ of an indexed family of sets $\{S_\alpha\}_A$ is the set of all functions $x:A \to \cup S_\alpha$ such that for each α, $x(\alpha) = x_\alpha$ belongs to S_α. The function x is often denoted by (x_α), or $\{x_\alpha\}$, the indexed collection of its values. In particular, when A is the set $\mathbb{N}$ of natural numbers, with generic element n, the function x is also called a "sequence", and often denoted (x_n) or $\{x_n\}$.

Definition: Let $\{X_\alpha\}_A$ be an arbitrary indexed collection of modules. A **(direct) product** of this family is a module X and a collection of homomorphisms $\pi_\alpha:X \to X_\alpha$ with the following property: For every module Y, the correspondence taking a homomorphism $f:Y \to X$ to the family of compositions $\{\pi_\alpha \circ f\}$ is a bijection between $\mathrm{Hom}(Y,X)$ and the cartesian product set $\prod_A \mathrm{Hom}(Y,X_\alpha)$.

Notation: A product of the $\{X_\alpha\}$, if one exists, is denoted variously as $\prod_A X_\alpha$, or $\prod_\alpha X_\alpha$, or $\prod X_\alpha$, etc....

The unique map $f:Y \to \prod X_\alpha$ which corresponds to the family $\{f_\alpha\}$ of maps $f_\alpha:Y \to X_\alpha$, may be denoted $f = \prod f_\alpha : Y \to \prod X_\alpha$. Thus $(\prod f_\alpha)(y) = \{f_\alpha(y)\}$.

Terminology: A direct product is also called simply a "product" or "categorical product", the modules X_α are called "direct factors" of $\prod X_\alpha$, the maps $\tau_\beta:\prod X_\alpha \to X_\beta$ are the "projections", and the maps $\tau_\alpha \circ f = f_\alpha$ are the "component maps" of the map $f = \prod f_\alpha$.

Remarks: (a) The notation for the direct product module of a family of modules is the same as the notation for the cartesian product of those modules as sets, and there are two justifications for this: first, the underlying set of the direct product module is the cartesian product set (see theorem below); second, the direct product of a family of sets, defined by the analogous mapping property, is exactly the cartesian product set.
(b) Intuitively the factors X_α may be thought of as submodules of $\prod X_\alpha$, onto which projections τ_α are given, and the definition above means (i) a map $f:Y \to X$ is determined by all its compositions $f_\alpha = \tau_\alpha \circ f:Y \to X_\alpha$ with these projections, and (ii) for any family of maps $\{f_\alpha:Y \to X_\alpha\}$, there is a (unique) map $f = \prod f_\alpha:Y \to X$ whose compositions are the f_α.

In the following theorem, take special note of the uniqueness proof, which has a characteristically "mapping theoretic" flavor.
Theorem: A direct product of any family $\{X_\alpha\}$ of modules exists, and is unique up to unique isomorphism.
proof: As a set, let $\prod X_\alpha$ be the cartesian product of the family $\{X_\alpha\}$. Define an R-module structure on the cartesian product (whose elements we recall are functions), "pointwise". Thus $\{x_\alpha\}+\{y_\alpha\} = \{x_\alpha+y_\alpha\}$ and $r\{x_\alpha\} = \{rx_\alpha\}$, just as for finite products.
The properties of an R-module are readily checked, eg. the identity is $\{0_\alpha\}$ and the inverse of $\{x_\alpha\}$ is $\{-x_\alpha\}$.
For every β in A, the map $\tau_\beta:\prod X_\alpha \to X_\beta$, takes the function $\{x_\alpha\}$ to x_β, its value at β. For each β, τ_β is a homomorphism by definition of the pointwise operations in $\prod X_\alpha$. For any $f:Y \to \prod X_\alpha$, and any point y in Y, knowing all the compositions $(\tau_\alpha \circ f)(y) = f(y)_\alpha$, tells us all the values of the function $(f(y)_\alpha)$. Hence the compositions $\{\tau_\alpha \circ f\}$ determine f. On the other hand, given a family of maps $\{f_\alpha:Y \to X_\alpha\}$, we can define $f:Y \to \prod X_\alpha$ as follows: for each y in Y, $f(y)_\alpha = f_\alpha(y)$. Then each composition $\tau_\alpha \circ f$ equals f_α. Moreover, since $f(y+z) = \{f(y+z)_\alpha\} = \{f_\alpha(y+z)\} = \{f_\alpha(y) + f_\alpha(z)\} = \{f_\alpha(y)\} + \{f_\alpha(z)\} = \{f(y)_\alpha\} +$

$\{f(z)_\alpha\} = f(y) + f(z)$, thus f is additive. Similarly, f is R-linear, hence a homomorphism. This proves *existence* of the product.

As for *uniqueness*, let X be another product, with projections $\pi_\alpha: X \to X_\alpha$. We want to show $X \cong \prod X_\alpha$. Following Auslander's dcitum, first we just look for a map each way, then try to show the maps are inverse isomorphisms. But a map into $\prod X_\alpha$ is determined by a family of maps into the X_α, precisely what the π_α give us! So, by the mapping property of $\prod X_\alpha$, the projections π_α determine a unique map $\varphi: X \to \prod X_\alpha$, whose compositions are $\varphi_\alpha = \tau_\alpha \circ \varphi = \pi_\alpha$. Similarly, the mapping property of X and the τ_α determine a unique map $\psi: \prod X_\alpha \to X$ whose compositions are $\psi_\alpha = \pi_\alpha \circ \psi = \tau_\alpha$. To show that φ and ψ are mutually inverse isomorphisms, of course we have to check that their compositions are both identities. Now the identity map $1: X \to X$ has the compositions $1_\alpha = \pi_\alpha$, and is the only map with those compositions. So we must check that $\psi \circ \varphi: X \to X$ also has $\pi_\alpha = (\psi \circ \varphi)_\alpha$. I.e. we must compute the compositions $(\psi \circ \varphi)_\alpha = (\pi_\alpha \circ \psi \circ \varphi): X \to X_\alpha$. By definition of ψ, $\pi_\alpha \circ \psi = \tau_\alpha$, and then by definition of φ, $\tau_\alpha \circ \varphi = \pi_\alpha$. Thus indeed $(\psi \circ \varphi)_\alpha = \pi_\alpha = 1_\alpha$, for all α. Hence $(\psi \circ \varphi) = 1$. Similarly, $(\varphi \circ \psi) = 1$. QED.

Remark: (a) The uniqueness proof clarifies the meaning of the phrase "unique isomorphism" in the theorem: more precisely, if $(X, \{\tau_\alpha\})$ and $(\tilde{X}, \{\tilde{\tau}_\alpha\})$ are both products of $\{X_\alpha\}$, the unique maps $\varphi: X \to \tilde{X}$ and $\psi: \tilde{X} \to X$ such that $\tilde{\tau}_\alpha \circ \varphi = \tau_\alpha$ and $\tau_\alpha \circ \psi = \tilde{\tau}_\alpha$ for all α, are mutually inverse isomorphisms.
(b) It is probably more common to denote the projections τ_α associated to the product $\prod X_\alpha$, by lower case π, i.e. $\pi_\alpha: \prod X_\alpha \to X_\alpha$, which makes the choice of letter easier to remember, but sometimes it is helpful to have more variety in the choice of letters.

Exercise #166) Define a (direct) product of a family $\{X_\alpha\}$ of sets by imitating the definition of a direct product of modules above, but substituting the word "set" for "module", "function" for "homomorphism", and substituting "Hom$\mathcal{S}$" (the family of set functions) for "HomR". Prove that the cartesian product set $\prod X_\alpha$ with the usual projections $\pi_\alpha: \{x_\alpha\} \mapsto x_\alpha$ is a product of the family $\{X_\alpha\}$, and that any other product is bijectively equivalent to it.

Next we prove the existence and uniqueness of *sums* of modules.

Definition: Let $\{X_\alpha\}$ be an indexed collection of modules. A **(direct) sum** of this family is a module X and a collection of homomorphisms $\sigma_\alpha : X_\alpha \to X$ with the following mapping property: For every module Y the correspondence taking a homomorphism $f: X \to Y$ to the family of compositions $\{f \circ \sigma_\alpha\}$ is a bijection between $\mathrm{Hom}(X,Y)$ and the cartesian product $\prod_A \mathrm{Hom}(X_\alpha, Y)$.

Notation: A direct sum of the family $\{X_\alpha\}_A$, if one exists, is denoted $\oplus_A X_\alpha$, $\oplus_\alpha X_\alpha$, or $\oplus X_\alpha$, and also $\amalg_A X_\alpha$, $\amalg_\alpha X_\alpha$, or $\amalg X_\alpha$; and the unique map $f: \oplus X_\alpha \to Y$ corresponding to the family $\{f_\alpha\}$ may be denoted $f = \oplus f_\alpha : \oplus X_\alpha \to Y$, or $f = \amalg f_\alpha : \amalg X_\alpha \to Y$

Terminology: A direct sum is also called a "sum", or "categorical sum", the modules X_α are "direct summands" of $\oplus X_\alpha$, the maps $\sigma_\beta : X_\beta \to \oplus X_\alpha$ are the "injections", and the compositions $f_\alpha = f \circ \sigma_\alpha : X_\alpha \to Y$ associated to the map $f: \oplus X_\alpha \to Y$, are the "component maps" of f.

Remarks: **(a)** Intuitively, the summands X_α are thought of as submodules of $\oplus X_\alpha$, (although they are really only isomorphic to submodules) and the definition above means (i) a map $X \to Y$ is determined by its restrictions to all the X_α, and (ii) those restrictions can chosen arbitrarily on the X_α.
(b) Sometimes one sees the terms "internal" or "external" direct sums, depending on whether the maps $\sigma_\alpha : X_\alpha \to \oplus X_\alpha$ are inclusions or only injections, i.e. whether the X_α are actually submodules of $\oplus X_\alpha$ or only isomorphic to submodules. From the mapping theoretic point of view this distinction becomes very minor: i.e. we tend on the one hand to identify two objects if a specific isomorphism is given between them, and on the other hand even if $M \subset N$ is a submodule we think of the inclusion as a map from M to N. Hence inclusion is just another map, with no more special properties than any other injective homomorphism. The usefulness of the new point of view is in unifying concepts and generalizing, while the power of the old is its concreteness and familiarity, hence sometimes making computations seem easier.

Theorem: A direct sum of any family $\{X_\alpha\}_A$ of modules exists, and is unique up to unique isomorphism.

proof: Consider the direct product ΠX_α of the modules X_α, and then let $\oplus X_\alpha \subset \Pi X_\alpha$ be the subset consisting of those functions $x = \{x_\alpha\}$ such that $x_\alpha = 0$ for all but a finite number of α. This set is closed under addition and scalar multiplication, hence forms a submodule. We claim this submodule, together with the following natural injections $\{\sigma_\alpha\}$, is a direct sum of $\{X_\alpha\}$. For each β, the map $\sigma_\beta\, X_\beta \to \oplus X_\alpha$ takes z in X_β to the function $\{x_\alpha\}$ such that $x_\beta = z$, and for $x_\alpha = 0$ for $\alpha \neq \beta$. Since $\sigma_\beta(z)$ has only one non zero value, it belongs to the submodule $\oplus X_\alpha$. The pointwise definition of the R-module structure on ΠX_α shows σ_β is a module map. If $f,g:\oplus X_\alpha \to Y$ are module maps such that the compositions $f_\alpha = f\circ\sigma_\alpha = g\circ\sigma_\alpha = g_\alpha$, are equal for all α, then let $\{x_\alpha\}$ be any element of $\oplus X_\alpha$, and let $\beta,\gamma,\ldots\delta$ be the indices corresponding to non zero values of $\{x_\alpha\}$. Then $\{x_\alpha\} = \sigma_\beta(x_\beta)+\sigma_\gamma(x_\gamma)+\ldots+\sigma_\delta(x_\delta)$, so
$f(\{x_\alpha\}) = (f\circ\sigma_\beta)(x_\beta)+(f\circ\sigma_\gamma)(x_\gamma)+\ldots+(f\circ\sigma_\delta)(x_\delta) =$
$(g\circ\sigma_\beta)(x_\beta)+(g\circ\sigma_\gamma)(x_\gamma)+\ldots+(g\circ\sigma_\delta)(x_\delta) = g(\{x_\alpha\})$. Thus the compositions $f\circ\sigma_\alpha$ determine f.

Then, if we are given any collection of maps $\{f_\alpha:X_\alpha \to Y\}$, we define a function $f = \oplus f_\alpha:\oplus X_\alpha \to Y$ by setting $f(\{x_\alpha\}) = \Sigma f_\alpha(x_\alpha)$. In this infinite sum, all but a finite number of terms are zero, and we define this "sum" to mean the sum of the non zero terms. Then $f(\{x_\alpha\}+\{y_\alpha\}) = f(\{x_\alpha+y_\alpha\}) = \Sigma f_\alpha(x_\alpha+y_\alpha) = \Sigma[f_\alpha(x_\alpha)+f_\alpha(y_\alpha)] = \Sigma f_\alpha(x_\alpha) + \Sigma f_\alpha(y_\alpha) = f(\{x_\alpha\}) + f(\{y_\alpha\})$. (These sums make sense because in all cases only a finite number of terms are non zero.) Similarly, $f = \oplus f_\alpha$ is R-linear.

Finally, for each β the composition $(\oplus f_\alpha\circ\sigma_\beta):X_\beta \to Y$ has at z the value $(\oplus f_\alpha\circ\sigma_\beta)(z) = \Sigma_{\alpha\neq\beta}\, f_\alpha(0) + f_\beta(z) = f_\beta(z)$. Thus for all β, $(\oplus f_\alpha\circ\sigma_\beta) = f_\beta$.

This proves *existence* of a sum.

The *uniqueness* proof is dual to that for a product. I.e. if $(X,\ \rho_\alpha\, X_\alpha \to X)$ is another sum, the family of maps $\{\rho_\alpha:X_\alpha \to X\}$ determines a unique map $\varphi\,\oplus X_\alpha \to X$ such that $\varphi\circ\sigma_\alpha = \rho_\alpha$ for all α, while the family $\{\sigma_\alpha:X_\alpha \to \oplus X_\alpha\}$ determines a unique map $\psi:X \to \oplus X_\alpha$ such that $\psi\circ\rho_\alpha = \sigma_\alpha$ for all α. To show that $\varphi\circ\psi$ and $\psi\circ\varphi$ are identities, it suffices to compute the component maps $\varphi\circ\psi\circ\rho_\alpha$ and $\psi\circ\varphi\circ\sigma_\alpha$. By definition of the φ,ψ, we have $\varphi\circ\psi\circ\rho_\alpha = \varphi\circ\sigma_\alpha = \rho_\alpha$.

and $\psi\circ\varphi\circ\sigma_\alpha = \psi\circ\rho_\alpha = \sigma_\alpha$. Since $\varphi\circ\psi$ and $\psi\circ\varphi$ have the same component maps as the identity maps, they are the identity maps. **QED.**

The whole point: (a) If $\{X_\alpha\}$ is a family of R-modules, and Y is any R-module, then the map taking f to the family $\{f\circ\sigma_\alpha\}$ is a bijection of the sets $\text{Hom}(\oplus X_\alpha, Y) \approx \prod \text{Hom}(X_\alpha, Y)$.
Dually, the map taking f to the family $\{\pi_\alpha\circ f\}$ is a bijection of the sets $\text{Hom}(Y, \prod X_\alpha) \approx \prod \text{Hom}(Y, X_\alpha)$
(b) Moreover, all four sets of maps in (a) are naturally R-modules, and the natural bijections above are R-module maps, hence yield module isomorphisms $\text{Hom}(\oplus X_\alpha, Y) \cong \prod \text{Hom}(X_\alpha, Y)$, and $\text{Hom}(Y, \prod X_\alpha) \cong \prod \text{Hom}(Y, X_\alpha)$, as well as equivalences of the corresponding functors from $\mathfrak{M}$ to $\mathfrak{M}$.

Remark: If the family of modules $\{X_\alpha\} = \{X_1, \ldots, X_n\}$ is finite, the product $\prod_{\alpha=1,\ldots,n} X_\alpha$ and the sum $\oplus_{\alpha=1\ldots n} X_\alpha$ are essentially the same, in particular the modules are isomorphic, but we focus on different maps, i.e. maps in or maps out. Even the modules are essentially different if the family is infinite. For other types of objects, not modules, sums and products may differ, even for finite families. For example, this occurs in the category of sets. First define a "disjoint union" operation for sets. If $A = \{1,2,3\}$ and $B = \{2,3,4,5\}$, then to form the disjoint union of these two sets, it is common to say "paint the elements of one set blue" to distinguish them from the elements of the other. This can be made precise, as follows. Choose two different objects $\alpha \neq \beta$, replace the set A by the set $A\times\{\alpha\} = \{1\times\{\alpha\}, 2\times\{\alpha\}, 3\times\{\alpha\}\}$, and replace B by $B\times\{\beta\} = \{2\times\{\beta\}, 3\times\{\beta\}, 4\times\{\beta\}, 5\times\{\beta\}\}$. The labels α,β are the two different "colors", distinguishing the elements of A from those of B. Just make sure, even if the sets A, B are the same, that the labels α,β you use are different. Then the new sets $A\times\{\alpha\}$ and $B\times\{\beta\}$ are disjoint, but bijectively equivalent to the original sets. Then the "disjoint union" $A \amalg B = (A\times\{\alpha\})\cup(B\times\{\beta\})$ of the original sets is defined to be the union of the new disjoint versions of those sets. This plays the role of a sum for the sets A,B

Exercise #167) Define a direct sum of a family $\{X_\alpha\}$ of sets by imitating the definition of a direct sum of modules above, but substituting the word "set" for "module", "function" for "homomorphism", and substituting "Hom$_\mathcal{S}$" [the family of set functions] for "Hom$_R$". Prove that if $\{X_\alpha\}$ is a collection of pairwise disjoint sets, their union $\cup X_\alpha$ together with the inclusions $\sigma_\alpha : X_\alpha \subset \cup X_\alpha$ is a sum of the family $\{X_\alpha\}$, and that any other sum is bijectively equivalent to it. [In general, the sum of an arbitrary family of sets is their *disjoint* union.]

Remark: The previous two exercises show that even for two disjoint sets A,B, the sum $A \amalg B$ and the product $A \times B$ are different. In the category of groups, the product of two copies of $\mathbf{Z}$ is the usual cartesian product $\mathbf{Z} \times \mathbf{Z}$ with its pointwise structure, isomorphic to the free Abelian group $Frab(\alpha,\beta)$ on two generators, but the sum $\mathbf{Z} \amalg \mathbf{Z}$ is the (non abelian) free group $Fr(\alpha,\beta)$ on two generators. If G,H are any groups, it is easy to show $G \times H$ with the usual pointwise operations is their product. Can you construct a *sum* of G,H? More generally, is there a sum of an arbitrary collection of groups?

Exercise #16B) If A is any set, consider the indexed family $\{R_\alpha\}_A$ where $R_\alpha \cong R$ for each α. Prove that $\oplus_A R_\alpha$ = {all functions $\{x_\alpha\}: A \to R$, with only a finite number of non zero values}, also denoted $\oplus_A R$, is a free R-module with basis $\{e_\alpha\}_A$, where e_β is the function $e_\beta: A \to R$ such that $e_\beta(\beta)=1$, but $e_\beta(\alpha) = 0$ for $\alpha \neq \beta$.

Remarks: **(a)** It follows from the previous exercise that for any set A there is an R-module which is "free on A", namely $\oplus_A R$. In particular $\oplus_A \mathbf{Z} \cong Frab(A)$, free abelian group on the set A.
(b) As a special case of the defining property of a sum, if $M \cong \oplus_A R$ is the free R-module on the set A, then $M^* \cong Hom(\oplus_A R,R) \cong \prod_A Hom(R,R) \cong \prod_A R$. In particular, $(\oplus_A R)^* \cong \prod_A R$.
(c) More generally, if $\{X_\alpha\}$ is any collection of modules, $(\oplus X_\alpha)^* = Hom(\oplus X_\alpha, R) \cong \prod Hom(X_\alpha,R) = \prod (X_\alpha)^*$, i.e. <u>the dual of a sum is the product of the duals</u>.
(d) One can show that the rank of an arbitrary free module is well defined, i.e. for any two sets A,B, $\oplus_A R \cong \oplus_B R$ iff $A \approx B$. [cf. appendix, and the section on tensor products.]

Question: We know $\oplus A R$ is a free module, but is the product $\prod A R$

$\cong (\oplus_A R)^*$ also free? It *is* free if R is a field, of course, but without an explicit basis, we need another approach to this question over R.

Natural Equivalence of Functors

For a general R module X, we know the module of homomorphisms $X^* = \mathrm{Hom}(X,R)$ *may* be much larger or much smaller than X, and in particular X is not always determined by $\mathrm{Hom}(X,R) = X^*$ [cf. ex. #151]. Since the mapping theoretic point of view in studying X says that we should focus on how we define maps in or out of X, it is important to observe that nonetheless X *is* completely determined by the maps of X into *everything!* I.e. given a module X, the *functor* $\mathrm{Hom}_R(X,\cdot)$ does determine X. To see this, we review the concept of functor, and then describe an appropriate notion of equivalence.

Recall that a **(covariant) functor** $F:\mathfrak{M}\to\mathfrak{M}$ assigns to each module M another module F(M), and to each module map $\alpha:M\to N$, a corresponding module map $F(\alpha):F(M)\to F(N)$. It is further required that identities go to identities, $F(1_M) = 1_{F(M)}$, and compositions go to compositions $F(f\circ g) = F(f)\circ F(g)$. Covariant functors $F:\mathfrak{M}\to\mathcal{S}$ are defined similarly, (as well as contravariant functors).

Examples: The constructions of sums and products define functors, in the following sense: if $\{X_\alpha\}$ is a family of modules, the functor Π assigns to it the product module $(\Pi X_\alpha, \{\tau_\alpha:\Pi X_\alpha\to X_\alpha\}) = (\Pi X_\alpha,\{\tau_\alpha\})$. Given two families of modules $\{X_\alpha\}$, $\{\tilde{X}_\alpha\}$, if $\{f_\alpha:X_\alpha\to\tilde{X}_\alpha\}$ is a corresponding family of maps, then the family of compositions $\{f_\alpha\circ\tau_\alpha:\Pi X_\alpha\to\tilde{X}_\alpha\}$ determines a unique map $\Pi(f_\alpha\circ\tau_\alpha):\Pi X_\alpha\to\Pi\tilde{X}_\alpha$, between the two products. Thus Π is a functor from families of modules and maps (with a fixed index set A), to single modules and maps. We usually write simply Πf_α instead of $\Pi(f_\alpha\circ\tau_\alpha)$, for the map $\Pi X_\alpha\to\Pi\tilde{X}_\alpha$, in spite of a slight conflict with the previous use of that notation. [This is sometimes called "abuse of notation", and very handy it is too.]

Similarly, $\oplus$ is a functor from families of modules and maps to single modules and maps. I.e the functor $\oplus$ takes $\{f_\alpha:X_\alpha\to\tilde{X}_\alpha\}$ to $\oplus f_\alpha:\oplus X_\alpha\to\oplus\tilde{X}_\alpha$.

Definition: Two (covariant) functors F,G are called "equivalent",

$F \cong G$, or "**naturally equivalent**" if there exist isomorphisms $\varphi_M:F(M)\to G(M)$, one for each module M, such that for each map $\alpha:M\to N$, the induced maps $F(\alpha):F(M)\to F(N)$ and $G(\alpha):G(M)\to G(N)$ are identified by the isomorphisms φ_M and φ_N. I.e. the following two compositions are equal: $(G(\alpha)\circ\varphi_M) = (\varphi_N\circ F(\alpha)):F(M)\to G(N)$.

Remarks: **(a)** Equivalence of functors is an equivalence relation.
(b) Equivalence of *contravariant* functors is defined analogously.
(c) If we drop the requirement that the maps φ_M be isomorphisms, then the resulting family of φ_M is called a "natural transformation" of the functors F,G.
(d) Peter Freyd emphasized that natural transformations are an essential concept, asserting that the only reason we define categories is to define functors, and the only reason we define functors is to be able to define natural transformations.

Example: If $\Theta:M\to N$ is an isomorphism, then the two functors Hom(N,·) and Hom(M,·) are equivalent. I.e. for each X, the map Θ^* Hom(N,X)→Hom(M,X), "preceding by Θ", is an isomorphism, since it has as inverse $(\Theta^*)^{-1} = (\Theta^{-1})^*$. Secondly, if $f:X\to Y$ is a map, then the associated maps f_*:Hom(N,X)→Hom(N,Y), and f_*:Hom(M,X)→Hom(M,Y) are identified by Θ^*. That is, both compositions $(f_*\circ\Theta^*)$:Hom(N,X)→Hom(M,X)→Hom(M,Y), and $(\Theta^*\circ f_*)$:Hom(N,X)→Hom(N,Y)→Hom(M,Y), are equal. This is true by associativity of composition, since for $g:N\to X$, $(f_*\circ\Theta^*)(g) = f\circ(g\circ\Theta) = (f\circ g)\circ\Theta = (\Theta^*\circ f_*)(g)$.

Our goal is the following converse assertion:
Theorem: Suppose the functors Hom(N,·) and Hom(M,·) are equivalent, via the isomorphisms φ_X:Hom(N,X)→Hom(M,X), for all modules X. Then N and M are isomorphic via a unique map $\Theta:M\to N$ such that for all X, $\varphi_X = \Theta^*$.
proof: How do we prove something so abstract and complicated? As usual, just look for a natural map $\Theta:M\to N$, and then try to show it has an inverse. Now the only entirely naturally given maps are the identities $1_N:N\to N$, and $1_M:M\to M$. But we have by hypothesis an isomorphism φ_N:Hom(N,N)→Hom(M,N). This means we can transfer 1_N over to a map $\Theta = \varphi_N(1_N):M\to N$. Since this is the only map

obtainable from the assumptions, this must be it! [Another way to guess the map Θ, is that if indeed $\varphi X = \Theta^*$, then we can recover Θ as $\varphi N(1_N) = \Theta^*(1_N) = 1_N \circ \Theta = \Theta$. In particular this shows Θ is unique.]

The inverse map of course, should be $\varphi M^{-1}(1_M): N \to M$. To show these are indeed inverses, we must show their compositions are identities, using our hypotheses. The only hypotheses we have are those guarenteeing compatibility of φX and f_*. That is, for each $f: X \to Y$, the two compositions:

$(f_* \circ \varphi X): \text{Hom}(N,X) \to \text{Hom}(M,X) \to \text{Hom}(M,Y)$, and

$(\varphi Y \circ f_*): \text{Hom}(N,X) \to \text{Hom}(N,Y) \to \text{Hom}(M,Y)$, are equal. Let's apply this to the case $f = \varphi M^{-1}(1_M): N \to M$. Then we have equality of the two compositions: $(\varphi M^{-1}(1_M)_* \circ \varphi N): \text{Hom}(N,N) \to \text{Hom}(M,N) \to \text{Hom}(M,M)$, and $(\varphi M \circ \varphi M^{-1}(1_M)_*): \text{Hom}(N,N) \to \text{Hom}(N,M) \to \text{Hom}(M,M)$. Applying the first of these compositions to the element 1_N gives $(\varphi M^{-1}(1_M)_* \circ \varphi N)(1_N) = \varphi M^{-1}(1_M) \circ (\varphi N(1_N))$. Applying the second composition gives $(\varphi M \circ \varphi M^{-1}(1_M)_*)(1_N) = \varphi M(\varphi M^{-1}(1_M)) = 1_M$. This proves that $\varphi M^{-1}(1_M)$ is left inverse to $(\varphi N(1_N))$. The proof it is also right inverse is similar.

To see that $\varphi N(1_N)^*(f) = \varphi X(f)$ for all f in $\text{Hom}(N,X)$, consider the two equal compositions: $(\varphi X \circ f_*): \text{Hom}(N,N) \to \text{Hom}(N,X) \to \text{Hom}(M,X)$, and $(f_* \circ \varphi N): \text{Hom}(N,N) \to \text{Hom}(M,N) \to \text{Hom}(M,X)$. Tracing 1_N around both ways gives first $(\varphi X \circ f_*)(1_N) = \varphi X(1_N \circ f) = \varphi X(f)$, and then $(f_* \circ \varphi N)(1_N) = f_*(\varphi N(1_N)) = f \circ \varphi N(1_N) = (\varphi N(1_N))^*(f)$. QED.

Remarks: (a) We never used in this theorem that the maps φN were *module* isomorphisms, i.e. it suffices to assume they are bijections of sets. In fact, if X,Y are objects in *any* category $\mathfrak{C}$, such that $\text{Hom}_{\mathfrak{C}}(X,\cdot)$ and $\text{Hom}_{\mathfrak{C}}(Y,\cdot)$ are equivalent as functors $\mathfrak{C} \to \mathcal{S}$, then X and Y are isomorphic in $\mathfrak{C}$. The proof is the same. [Try it.]
(b) If the *contravariant* functors $\text{Hom}(\cdot,X)$ and $\text{Hom}(\cdot,Y)$ are equivalent, by the analogous definition, then we also have $X \cong Y$.
(c) These theorems mean that a module, or an object in any category, is uniquely characterized if we tell in a natural way, how to define all maps either into it or out of it, i.e. an object is determined by its "mapping properties".
(d) The general principle that every natural transformation $\{\varphi X\}$ of

Hom functors is induced by a map Θ of the corresponding modules, is often called "Yoneda's lemma". It follows from the last two sentences of the proof of the previous theorem.

Let's check next that the defining property of a direct sum, emphasized in "the whole point" above, is an equivalence of functors.

Theorem: For any family $\{X_\alpha\}$, the functors $Hom(\oplus X_\alpha,\cdot)$ and $\prod Hom(X_\alpha,\cdot)$ are equivalent.

proof: If $\{\sigma_\alpha:X_\alpha\to\oplus X_\alpha\}$ are the canonical injections, then for each Y, we know the map $\varphi_Y = \prod\sigma_\alpha{}^*: Hom(\oplus X_\alpha,Y)\to\prod Hom(X_\alpha,Y)$ taking f to $\{f\circ\sigma_\alpha\}$ is a bijection. So we must check that if $g:Y\to Z$ is any map, then the following two compositions are equal:

i) $\prod g_*\circ\varphi_Y = \prod g_*\circ\prod\sigma_\alpha{}^*: Hom(\oplus X_\alpha,Y)\to\prod Hom(X_\alpha,Y)\to\prod Hom(X_\alpha,Z)$,

and ii) $\varphi_Z\circ g_* = \prod\sigma_\alpha{}^*\circ g_*.Hom(\oplus X_\alpha,Y)\to Hom(\oplus X_\alpha,Z)\to\prod Hom(X_\alpha,Z)$.

To check it, if $f:\oplus X_\alpha\to Y$ is any element of the left hand module, then the first composition yields $\prod g_*(\{f\circ\sigma_\alpha\}) = \{g\circ f\circ\sigma_\alpha\}$, while the second gives $(\prod\sigma_\alpha{}^*)(g\circ f) = \{g\circ f\circ\sigma_\alpha\}$, the same result. **QED.**

Corollary: The previous two theorems, allow a very short version of the proof that direct sums are unique up to isomorphism.

proof: If X,Y are both sums of the family $\{X_\alpha\}$ there are equivalences $Hom(X,\cdot)\cong\prod Hom(X_\alpha,\cdot)\cong Hom(Y,\cdot)$. Thus $X\cong Y$. **QED.**

Remarks: (i) The argument for the previous corollary does not display the isomorphism $X\cong Y$, but it would do so if we were more careful in describing the equivalences of functors which induce it. I.e. put X into the second variable of every functor in the last line of the proof), and trace 1_X through the equivalences from left to right. Then 1_X goes to the family $\{\sigma_\alpha\}$ in $\prod Hom(X_\alpha,X)$, of injections associated to the sum X. Then these correspond to the unique map $f:Y\to X$ such that for each injection $\tilde\sigma_\alpha X_\alpha\to Y$, $f\circ\tilde\sigma_\alpha = \sigma_\alpha$. As you might expect, this is the same isomorphism we found before when proving any two sums of the same family are isomorphic. That f is an isomorphism follows here from the fact that it is induced by equivalences of functors, with f^{-1} obtained by putting Y in the second variable in each functor and tracing 1_Y in the other direction, so $f^{-1}=$ the map $g:X\to Y$ such that $g\circ\sigma_\alpha = \tilde\sigma_\alpha$.

(ii) Whenever we say two modules are isomorphic, we should ask whether we can specify a particular isomorphism. If all we know is that they are isomorphic, then we can only conclude general things about them, such as: if one is free then so is the other and they have the same rank; also they have the same annihilator, and so on. If we know an actual isomorphism, we can do much more, such as actually replace one of them by the other in any sequence of maps. We can also identify specific elements of one with specific corresponding elements of the other. We can really equate them by setting up a dictionary for how to replace elements of one by elements of the other. Some of us used to tell beginning students that "isomorphic" objects could be considered as "the same", but this is not really true. Two objects can only be considered as fully interchangeable if we are given a specific isomorphism, by means of which they are to be identified.
(iii) In almost all situations in this section, the various isomorphic objects are isomorphic by means of possibly many different isomorphisms, but generally there is one *distinguished* choice of isomorphism, and that is the one we almost always want to use. If this were not true, we might get in trouble when we compose isomorphisms, since a composition of arbitrary isomorphisms would yield an isomorphism of the first and last objects in the chain, and it might differ from the isomorphism we had already chosen for them. Roughly, a choice of a family of isomorphisms which always compose to give the expected result is called a "coherent" family of isomorphisms. We expect that if we choose ours in the most obvious way, they will always be coherent. For instance, in the proof of the corollary above, we knew one isomorphism of the two sums $X \cong Y$ from our ealier proof, which thus induced an equivalence of functors $Hom(X,\cdot) \cong Hom(Y,\cdot)$ On the other hand we also had the chain of equivalences $Hom(X,\cdot) \cong \prod Hom(X_\alpha,\cdot) \cong Hom(Y,\cdot)$, whose composition provided another equivalence of those functors. Life is much simpler because those equivalences are the same.

Exercise #169) Prove a similar theorem and corollary for products. Show what the isomorphism is, as we did in remark (i).

Representable Functors

We know the covariant functors $Hom(X,\cdot)$ "commute with direct products", in the sense that $Hom(X,\prod Y_\alpha) \cong \prod Hom(X,Y_\alpha)$, and the contravariant functors Hom(·,X) change sums into products, i.e. $\mathrm{Hom}(\oplus Y_\alpha, X) \cong \prod \mathrm{Hom}(Y_\alpha, X)$. Since Hom functors are so important, we would like to know if they have any *other* characteristic properties. Ideally we would like to recognize a Hom functor from its properties. Thus we ask, of a functor $F:\mathfrak{M}\to\mathfrak{M}$, what properties would guarantee that F is equivalent to a Hom functor?

Definition: A functor $F:\mathfrak{M}\to\mathfrak{M}$ such that for some X, either $F \cong \mathrm{Hom}(X,\cdot)$, or $F \cong \mathrm{Hom}(\cdot,X)$, is called a "representable" functor.

Remark: If F is representable, we know the representing object X is unique up to isomorphism.

Properties of Hom Functors

Consider the covariant Hom functors Hom(X,·). The property of commuting with direct products is, as we have stated it above, a property of these functors acting on *objects*. Since we know maps are even more important than objects, we ask what Hom(X,·) does to products of *maps*. You showed in exercise #158) that the isomorphisms $\mathrm{Hom}(X,\prod Y_\alpha) \cong \prod \mathrm{Hom}(X,Y_\alpha)$ are natural in X, i.e. that the functors $\mathrm{Hom}(\cdot,\prod Y_\alpha) \cong \prod \mathrm{Hom}(\cdot,Y_\alpha)$ are equivalent, and we point out now that the isomorphisms are also natural in the Y_α. I.e. $\mathrm{Hom}(X,\prod_A(\cdot)_\alpha)$, and $\prod_A \mathrm{Hom}(X,(\cdot)_\alpha)$ are equivalent functors from families $\{f_\alpha:Y_\alpha\to Z_\alpha\}$ of modules and maps indexed by A, to single modules and single maps, via the isomorphisms $\varphi_Y:\mathrm{Hom}(X,\prod Y_\alpha) \cong \prod \mathrm{Hom}(X,Y_\alpha)$. I.e. given $\{f_\alpha:Y_\alpha\to Z_\alpha\}$, the isomorphisms φ_Y and φ_Z identify the map $\prod(f_{\alpha *}):\prod \mathrm{Hom}(X,Y_\alpha)\to\prod\mathrm{Hom}(X,Z_\alpha)$ with the map $(\prod f_\alpha)_*:\mathrm{Hom}(X,\prod Y_\alpha)\to\mathrm{Hom}(X,\prod Z_\alpha)$. Thus $\prod(f_{\alpha *})\circ\varphi_Y = \varphi_Z\circ(\prod f_\alpha)_*$, and in that sense "lower star" commutes with products (of maps). In the case of a family $\{f_\alpha:Y\to Z_\alpha\}$ inducing the map $\prod f_\alpha:Y\to\prod Z_\alpha$, the isomorphism $\varphi_Z:\mathrm{Hom}(X,\prod Z_\alpha)\cong\prod\mathrm{Hom}(X,Z_\alpha)$ identifies the maps $\prod(f_{\alpha *}):\mathrm{Hom}(X,Y)\to\prod\mathrm{Hom}(X,Z_\alpha)$ and $(\prod f_\alpha)_*:\mathrm{Hom}(X,Y)\to\mathrm{Hom}(X,\prod Z_\alpha)$. I.e. $\varphi_Z\circ(\prod f_\alpha)_* = \prod(f_{\alpha *})$, a similar "commutativity" result.

Left Exactness

Let's look further at the behavior of the operation "lower star" taking f to f_*. A natural question is whether it preserves the standard properties of maps. All functors preserve isomorphisms, so

we ask what happens to injective maps? I.e. if $f:X \to Y$ is an injection, what about f_*? Since these are module maps, it suffices to show if $g \neq 0$, where $g:Z \to X$, then $f_*(g) \neq 0$. If $g \neq 0$, there is some z such that $g(z) \neq 0$. Then $(f_*(g))(z) = (f \circ g)(z) = f(g(z)) \neq 0$, since f is injective. Thus $f_*(g) \neq 0$, and f_* is injective. Thus lower star does preserve injectivity. More precisely, if $f:X \to Y$ is injective, then for every Z, $f_*:\mathrm{Hom}(Z,X) \to \mathrm{Hom}(Z,Y)$ is injective.
What about surjections? If $f:X \to Y$ is surjective, is f_* always surjective? I.e. for every choice of Z, does every map $h:Z \to Y$ have the form $f_*(g) = (f \circ g)\ Z \to X \to Y$? This is the so called "lifting problem", and it does not always have a solution. For instance, if $f:\mathbf{Z} \to \mathbf{Z}_n$ is the (surjective) canonical projection, where $n \geq 2$, and $h:\mathbf{Z}_n \to \mathbf{Z}_n$ is the identity map, then h does not equal any composition $(f \circ g):\mathbf{Z}_n \to \mathbf{Z} \to \mathbf{Z}_n$, since the only map $g:\mathbf{Z}_n \to \mathbf{Z}$ is identically zero. Thus for every g, $f_*(g) = f \circ g = f \circ 0 = 0 \neq h$, f_* is not surjective although f is, so lower star does not always preserve surjectivity. The strongest possible true statement generalizing the injectivity - preserving property, is called "left exactness".

Definition: A covariant functor $F:\mathfrak{M} \to \mathfrak{M}$ is "left exact" iff whenever $0 \to A \to B \to C$ is an exact sequence of modules, then the associated sequence of modules $0 \to F(A) \to F(B) \to F(C)$ is also exact.

Remark: This definition says left exact functors "preserve kernels", in the sense that if $\alpha:A \to B$ is an embedding of A onto the kernel of $\beta.B \to C$, then $F(\alpha):F(A) \to F(B)$ is an embedding of $F(A)$ onto the kernel of $F(\beta):F(B) \to F(C)$.

Theorem: For any module X, $\mathrm{Hom}(X,\cdot)$ is left exact.
proof: If $0 \to A \to B \to C$ is an exact sequence of modules, we must show $0 \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C)$ is also exact. Since by the remarks above injections are preserved, it follows that $0 \to \mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B)$ is exact. Thus we only have to show exactness at $\mathrm{Hom}(X,B)$. We name the maps $\alpha:A \to B$, $\beta:B \to C$, so that we have $\alpha_*:\mathrm{Hom}(X,A) \to \mathrm{Hom}(X,B)$, and $\beta_*\ \mathrm{Hom}(X,B) \to \mathrm{Hom}(X,C)$. We must show $\ker(\beta_*) = \mathrm{Im}(\alpha_*)$. Since $(\beta_* \circ \alpha_*) = (\beta \circ \alpha)_* = (0)_* = 0$, we see that $\mathrm{Im}(\alpha_*) \subset \ker(\beta_*)$. Conversely, if f is in $\ker(\beta_*)$, i.e. if $\beta_*(f) = \beta \circ f = 0$, for some $f:X \to B$, then $\mathrm{Im}(f) \subset \ker(\beta) = \mathrm{Im}(\alpha) = \alpha(A) \subset B$, by exactness of $0 \to A \to B \to C$. Since α is injective, it gives an

R, and every map $f:M \to N$, we have $F(rf) = rF(f):F(M) \to F(N)$. In particular, if $r:M \to M$ denotes multiplication by r, then $F(r) = r:F(M) \to F(M)$; i.e. F(r) is also multiplication by r.

#173) Check that Hom(X,·) is a linear functor.

Representability Criteria

We know Hom(X,·) is a linear, left exact functor that commutes with direct products. I claim this *almost* characterizes covariant representable functors, and that all one needs further is the fact that they commute with "inverse limits", a generalization of direct products, discussed later. Analogously, F(·) = Hom(·,X) is the only linear functor F changing sums into products, cokernels into kernels, and such that $F(R) \cong X$. In particular, the dual functor $(\cdot)^* = \text{Hom}(\cdot,R)$ is the unique linear functor F changing sums into products, cokernels into kernels, and "fixing R", i.e. such that $F(R) \cong R$.

Let's indicate the proof for the dual functor:

Theorem: Assume $F:\mathfrak{M} \to \mathfrak{M}$ is linear, converts cokernels into kernels, sums into products, and $F(R) \cong R$. Then $F(\cdot) \cong \text{Hom}(\cdot,R)$.

proof sketch: We want to find, for each module X, a natural isomorphism $\varphi_X:X^* \to F(X)$. We know duality is well behaved on free modules, so first we represent X as a quotient of a free module in a canonical way: namely, there is a canonical surjective map $\oplus_X R \to X \to 0$, and if $K = K_X$ is its kernel, we can map another canonical free module onto K, to get an exact sequence of form $\oplus_K R \to \oplus_X R \to X \to 0$. Now apply both $(\cdot)^*$ and F(·) to this sequence and "simplify", to get the two sequences: $0 \to X^* \to \prod_X R \to \prod_K R$ and $0 \to F(X) \to \prod_X R \to \prod_K R$. the point is to check then that the two maps on the right ends of these sequences, the maps $\prod_X R \to \prod_K R$, are the same in both cases. Then it would follow by "uniqueness of kernels", that there is a natural isomorphism $\varphi_X:X^* \to F(X)$. This is what we claimed. Naturality of course must be checked. QED.

Remark: If in the previous theorem we assume $F(R) \cong Y$, the same proof shows $F(\cdot) \cong \text{Hom}(\cdot,Y)$.

Exercise #174): (Uniqueness of kernels, cokernels): In the commutative diagrams below, if the rows are exact, and the isomorphism from A to $\alpha(A)$, so can define $\alpha^{-1}:\alpha(A)\to A$. Since $f(X)\subset\alpha(A)$, then $g = (\alpha^{-1}\circ f):X\to A$ is defined, and for all z in X, $(\alpha_*(g))(z) = \alpha(\alpha^{-1}(f(z))) = f(z)$. Thus $\alpha_*(g) = f$, and $\ker(\beta_*)\subset \mathrm{Im}(\alpha_*)$. QED.

Exercise #170) Prove the converse of the previous theorem; i.e. a sequence $0\to A\to B\to C$ is exact iff for all X, the associated sequence $0\to \mathrm{Hom}(X,A)\to \mathrm{Hom}(X,B)\to \mathrm{Hom}(X,C)$ is exact. [In particular, f is injective iff for all X, $f_{*,X}$ is injective.]

Exercise #171) (i) Prove: If $A\to B\to C\to 0$ is exact, then for all X, $0\to \mathrm{Hom}(C,X)\to \mathrm{Hom}(B,X)\to \mathrm{Hom}(A,X)$ is exact. [In particular, if f is surjective then for all X, f^*_X is injective.]
(ii) Give an example of an exact sequence $A\to B\to 0$ such that $\mathrm{Hom}(X,A)\to \mathrm{Hom}(X,B)\to 0$ is not exact.

Exercise #172) If $0\to A\to B\to C\to 0$ is "split exact", i.e. if the map $B\to C$ has a right inverse inducing $B \cong A\oplus C$, then for every X, prove the sequences $0\to \mathrm{Hom}(X,A)\to \mathrm{Hom}(X,B)\to \mathrm{Hom}(X,C)\to 0$, and $0\to \mathrm{Hom}(A,X)\to \mathrm{Hom}(B,X)\to \mathrm{Hom}(C,X)\to 0$, are exact.

We prove the converse of part (i) the previous exercise:
Lemma: The sequence $A\to B\to C\to 0$ is exact iff for all X, $0\to \mathrm{Hom}(C,X)\to \mathrm{Hom}(B,X)\to \mathrm{Hom}(A,X)$ is exact. [In particular, f is surjective iff for all X, f^*_X is injective.]
proof: In view of Ex. 160, we prove "if", by contradiction. Denote the maps by $g:A\to B$, and $f:B\to C$.
(i) If $f:B\to C$ is not onto, then $h:C\to C/\mathrm{Im}(f) = X$, implies $h \neq 0$, but $f^*h = 0$, so $f^*:\mathrm{Hom}(C,X)\to \mathrm{Hom}(B,X)$ is not injective.
(ii) If $(f\circ g) \neq 0$, then $id:C\to C = X$ implies $(g^*\circ f^*)(id) = f\circ g \neq 0$, i.e. $(g^*\circ f^*) \neq 0$.
(iii) If $\mathrm{Im}(g)\subset\ker(f)$, but equality does not hold, then letting $h:B\to B/\mathrm{Im}(g) = X$ implies $g^*(h) = 0$. Thus h is in $\ker(g^*)$, but h does not vanish on $\ker(f)$, so h does not factor through $C \cong B/\ker(f)$, i.e. h is not in $\mathrm{Im}(f^*)$. QED.

There is one other basic property of Hom functors, "linearity".
Definition: A covariant functor $F:\mathfrak{M}\to\mathfrak{M}$ is "linear" iff for every r in

vertical maps are isomorphisms,

$$\begin{array}{ccc} 0 \to A \to B \to C & & C_1 \to B_1 \to A_1 \to 0 \\ \downarrow \quad \downarrow & & \downarrow \quad \downarrow \\ 0 \to X \to Y \to Z & & Z_1 \to Y_1 \to X_1 \to 0 \end{array}$$

prove there are unique maps $\alpha: A \to X$, $\alpha_1: A_1 \to X_1$ keeping the diagrams commutative, and that α, α_1 are isomorphisms.

In the next result, finding X is a little harder:

Theorem: Assume $F:\mathfrak{M} \to \mathfrak{M}$ is linear, left exact, and commutes with inverse limits. Then for some X, $F(\cdot) \cong \mathrm{Hom}(X,\cdot)$.

proof: [cf. Charles E. Watts, Proc. of AMS, 1960, p.5-8; also Samuel Eilenberg, Journal of Indian Math Soc, 1960.] Omitted.

§14) Tensor products

When we asked whether left exactness was enough to insure that a functor is a Hom functor, Ernie suggested composing two Hom functors to get a counterexample, since such a composition would still be left exact. Eg. if both F,G are left exact and f is an injection then G(f) is an injection, so $F(G(f)) = (F \circ G)(f)$ is also an injection. Let's look at the result of composing two Hom functors. If $F(\cdot) = \mathrm{Hom}(X,\cdot)$, and $G(\cdot) = \mathrm{Hom}(Y,\cdot)$, then $(F \circ G)(\cdot) = \mathrm{Hom}(X,\mathrm{Hom}(Y,\cdot))$. This might not look like a Hom functor, but let's check it against the Eilenberg-Watts theorem above. The composition of two linear functors is linear, and since both functors commute with inverse limits, so does the composition. Since we don't know exactly what that means, let's observe that this functor commutes with direct products: i.e $(F \circ G)(\prod_\alpha Z_\alpha) = \mathrm{Hom}(X,\mathrm{Hom}(Y,\prod_\alpha Z_\alpha)) \cong \mathrm{Hom}(X,\prod_\alpha \mathrm{Hom}(Y,Z_\alpha)) \cong \prod_\alpha \mathrm{Hom}(X,\mathrm{Hom}(Y,Z_\alpha))$. The point is that the functor $F \circ G$ does satisfy the characterizing properties of a Hom functor so it must be one!

Question: What is the module M (unique up to isomorphism) such that $\mathrm{Hom}(X,\mathrm{Hom}(Y,\cdot)) \cong \mathrm{Hom}(M,\cdot)$?

The answer to this question will follow from an appropriate reformulation of the iterated Hom functor. What does an element α of $(F \circ G)(Z) = \mathrm{Hom}(X,\mathrm{Hom}(Y,Z))$ look like anyway? It takes an element x of X and makes it act like a linear map on Y. Doesn't that sound familiar? Remember the inner product on a real inner product space M? It allows us to look at any x in M as a linear map on M, the map "product with x" denoted $\langle\cdot,x\rangle\ M \to \mathbb{R}$. So an inner product on M gives a map $M \to M^* = \mathrm{Hom}(M,\mathbb{R})$. The inner product has this property because it is a function of two variables, and when we fix one of them, it is linear in the other, i.e. the inner product is a "bilinear" function $\langle\cdot,\cdot\rangle\ M\times M \to \mathbb{R}$. On the other hand, if $\alpha: M \to M^*$ is any linear map, we can define a product $[\cdot,\cdot]: M\times M \to \mathbb{R}$ by setting $[x,y] = \alpha(x)(y)$. This won't be symmetric or positive definite, but it will be linear in each variable separately. In the same way if $\alpha(\cdot,\cdot): X\times Y \to Z$ is any bilinear function, then for each x in X, $\alpha(x,\cdot): Y \to Z$ will be linear, and so α gives a map from X to Hom(Y,Z). Let's write down what this suggests, precisely:

Definition: If X,Y,Z are modules and $X\times Y$ is the cartesian product, a function $\alpha: X\times Y \to Z$ is called "bilinear" iff for all $x, \tilde{x}$, in X, $y, \tilde{y}$, in Y, and r in R, the following relations hold in Z:
(i) $\alpha(x+\tilde{x},y) = \alpha(x,y)+\alpha(\tilde{x},y)$,
(ii) $\alpha(x,y+\tilde{y}) = \alpha(x,y)+\alpha(x,\tilde{y})$,
(iii) $\alpha(rx,y) = r\alpha(x,y) = \alpha(x,ry)$

Notation: The set of all bilinear functions, or bilinear maps, from $X\times Y \to Z$, will be denoted $\mathrm{Bil}(X\times Y,Z)$.

Remarks: **(i)** The definition says that if you fix the value of one variable, the resulting function is linear in the other, i.e. a function of two variables is bilinear iff it is linear in each variable separately. **(ii)** It follows from (i) that the sum of two bilinear functions is bilinear, and a scalar multiple of a bilinear function is bilinear. Hence the negative of a bilinear function is bilinear and the zero function is bilinear. Since addition of module-valued functions is associative and commutative, $\mathrm{Bil}(X\times Y,Z)$ forms an R- module. **(iii)** It also follows from (i) that if $\alpha\ X\times Y \to Z$ is bilinear, and $f: Z \to W$ is linear, then the composition $(f\circ\alpha): X\times Y \to W$ is bilinear. I.e. if f is linear, and α is bilinear, then $f_*(\alpha)$ is bilinear. Also f_* is linear, i.e. $f_*(\alpha+\beta) = f\circ(\alpha+\beta) = f\circ\alpha + f\circ\beta = f_*(\alpha) + f_*(\beta)$, and $f_*(r\alpha) = rf_*(\alpha)$. Hence $\mathrm{Bil}(X\times Y,\cdot)$ defines a functor $\mathfrak{M} \to \mathfrak{M}$, taking Z to $\mathrm{Bil}(X\times Y,Z)$, and taking $f: Z \to W$ to $f_*: \mathrm{Bil}(X\times Y,Z) \to \mathrm{Bil}(X\times Y,W)$. (We already know lower star preserves compositions and identities.)

The next observation is the main point for us:
Lemma: For any modules X,Y,Z, the assignment taking α to the map $x \mapsto \alpha(x,\cdot)$, is an isomorphism φ_Z:Bil(X×Y,Z)→Hom(X,Hom(Y,Z)). Moreover, the isomorphisms φ_Z define an equivalence of functors φ:Bil(X×Y,·) ≅ Hom(X,Hom(Y,·)).

Exercise #175): Prove the previous lemma.

Now we have reduced our problem of representing the functor Hom(X,Hom(Y,·)), to one of representing the functor Bil(X×Y,·).
Question: Is there a module M such that for all modules Z, there are compatible isomorphisms φ_Z:Bil(X×Y,Z)→Hom(M,Z)?

To attack this, as always, we start simply by trying to find an M such that there exists a natural map φ:Bil(X×Y,Z)→Hom(M,Z). To make things easier on ourselves, let's forget that the maps α in Bil are bilinear, and just ask for a module M such that any function α:X×Y→Z induces a linear map $\varphi(\alpha)$:M→Z. We know from the theory of free modules that for any set S, there is a module with this property with respect to functions out of S, namely the free module on the set S. So let's begin by taking $\tilde{M} = \oplus_{X\times Y} R$, the free module with basis X×Y. The map φ taking a bilinear function α:X×Y→Z to the induced homomorphism $\varphi(\alpha)$:$\oplus_{X\times Y} R$→Z, yields φ Bil(X×Y,Z)→Hom($\tilde{M}$,Z) This is a start. We know however that although φ is injective, it cannot be surjective, since *every* function out of X×Y, not just bilinear ones, correspond to homomorphisms on $\tilde{M}$. Which homomorphisms in Hom($\tilde{M}$,Z) correspond to bilinear functions on X×Y? If they are the ones that vanish on some submodule K of $\tilde{M}$, we could mod out by that submodule and get a quotient module whose linear maps will correspond one-one with bilinear maps! Fortunately, that is just what happens.

Recalling that the "characteristic function" χ_S of a set S is the function which equals one on S and is zero elsewhere, for each element (x,y) of X×Y let us denote by $\chi_{(x,y)}$ the function X×Y→R which equals one at (x,y) and is zero elsewhere. Thus $\chi_{(x,y)}$ is the basis element of $\oplus_{X\times Y} R$ corresponding to the element (x,y) of X×Y. [In the past we also denoted this by $e_{(x,y)}$.] Now we can answer the question just posed:

Definition: Let $K \subset \tilde{M}$ be the submodule generated by the functions $\{ \chi(x+\tilde{x},y) - \chi(x,y) - \chi(\tilde{x},y),\quad \chi(x,y+\tilde{y}) - \chi(x,y) - \chi(x,\tilde{y}),$ $\chi(rx,y) - r\chi(x,y),\quad \chi(x,ry) - r\chi(x,y) \}$, for all $x, \tilde{x}$ in X, all $y, \tilde{y}$ in Y, all r in R.

Consider the quotient module $M = \tilde{M}/K$, and the natural map $\pi: X \times Y \to M$, composed of the injection of $X \times Y$ onto the basis of $\tilde{M}$, followed by the canonical projection $\tilde{M} \to M$. We claim the linear maps out of M, i.e. the linear maps out of $\tilde{M}$ that vanish on K, are precisely those induced from bilinear functions on $X \times Y$.

Lemma: With the definitions above of K, M, and $\pi: X \times Y \to M$, for every Z, the map $\pi^*: \mathrm{Hom}(M,Z) \to \mathrm{Bil}(X \times Y, Z)$ sending $f: M \to Z$ to $(f \circ \pi): X \times Y \to Z$, is an isomorphism.

proof: First note that since $\pi(x,y) = \chi(x,y)$, the definition of the generators of K exactly makes π a bilinear map. Hence following π by a linear map gives a bilinear map, so π^* is well defined. Since $\pi^*(af+bg) = (af+bg)\circ\pi = a(f\circ\pi)+b(g\circ\pi) = a(\pi^*f)+b(\pi^*g)$, π^* is linear. On the other hand, if $\alpha: X \times Y \to Z$ is bilinear, and $\varphi(\alpha): \tilde{M} \to Z$ is the induced linear map, then for each basis element $\chi(x,y)$, we have $\varphi(\alpha)(\chi(x,y)) = \alpha(x,y)$. Since α is bilinear, $\alpha(x+\tilde{x},y) = \alpha(x,y)+\alpha(\tilde{x},y)$, so $\varphi(\alpha)(\chi(x+\tilde{x},y)) = \varphi(\alpha)(\chi(x,y)) + \varphi(\alpha)(\chi(\tilde{x},y))$, i.e. $\varphi(\alpha)$ maps the first generator of K above to zero in Z.
Moreover, $\alpha(rx,y) = r\alpha(x,y)$, so $\varphi(\alpha)(\chi(rx,y)) = r\varphi(\alpha)(\chi(x,y))$, and thus $\varphi(\alpha)$ sends the third generator of K above to zero as well. Similarly, $\varphi(\alpha)$ sends every generators of K to zero in Z, hence maps the submodule K to zero in Z. Thus for every bilinear map $\alpha: X \times Y \to Z$, $\varphi(\alpha): \tilde{M} \to Z$ induces a unique linear map $f_\alpha: M \to Z$, such that $\alpha = f_\alpha \circ \pi$. Thus the correspondences $f \mapsto \pi^*(f)$, and $\alpha \mapsto f_\alpha$, are mutually inverse isomorphisms between $\mathrm{Hom}(M,Z)$ and $\mathrm{Bil}(X \times Y, Z)$. **QED.**

Notation: The module M constructed above, had nothing to do with Z, and has the property given in the previous lemma for all Z, hence we denote it $M = X \otimes_R Y$, or simply $X \otimes Y$, if the ring R is known. The class in $X \otimes Y$ of the basis element $\chi(x,y)$ of $\tilde{M}$, is denoted $x \otimes y$

Terminology: The R-module $X \otimes_R Y = X \otimes Y$, is called the "tensor product" of the R-modules X and Y.

Important Remarks: (i) The notation $X \otimes Y$ unfortunately suggests that every element has the form $x \otimes y$. As we see from the definition however, the elements $x \otimes y$ only correspond to the basis elements of $\tilde{M}$, and $X \otimes Y$ is a quotient of $\tilde{M}$. Hence the elements $x \otimes y$ are only a generating set of $X \otimes Y$, and the general element has form $\Sigma a_j (x_j \otimes y_j)$. *Try not to forget this;* doing so is the basic mistake in the subject.
(ii) The statement of the previous lemma is the whole point to remember when dealing with linear maps out of the tensor product $X \otimes Y$: i.e. they are equivalent to bilinear maps out of $X \times Y$. Indeed this is essentially the only way we have to define maps on $X \otimes Y$. Since the elements $x \otimes y$ of $X \otimes Y$ are not independent, but only generators, it is not so straightforward to define a linear map on $X \otimes Y$ by telling what the map is supposed to do to those generators. We also have to check that our proposed map kills all the bilinear "relations" of form $(x+\tilde{x}) \otimes y - x \otimes y - \tilde{x} \otimes y$, $x \otimes (y+\tilde{y}) - x \otimes y - x \otimes \tilde{y}$, $(rx) \otimes y - r(x \otimes y)$, and $x \otimes (ry) - r(x \otimes y)$, used in defining the submodule K which was "modded out" above to form $X \otimes Y = \tilde{M}/K$. The way to do this is <u>always</u> to define first a function on $X \times Y$, $\alpha: X \times Y \to Z$, then check α is bilinear, and finally deduce that α induces a unique linear map $\tilde{\alpha}: X \otimes Y \to Z$ such that $\tilde{\alpha}(x \otimes y) = \alpha(x,y)$, for every (x,y) in $X \times Y$.

Corollary: The module $X \otimes Y$ represents the equivalent functors $\mathrm{Hom}(X, \mathrm{Hom}(Y,\cdot)) \approx \mathrm{Bil}(X \times Y, \cdot)$. I.e. the universal bilinear function $\pi: X \times Y \to X \otimes Y$ induces compatible isomorphisms for all Z, $\pi^*: \mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Bil}(X \times Y, Z)$.
proof: Suppose $g: Z \to W$ is a linear map. We must show the isomorphisms $\pi^*: \mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Bil}(X \times Y, Z)$ and $\pi^*: \mathrm{Hom}(X \otimes Y, W) \to \mathrm{Bil}(X \times Y, W)$, are compatible with the maps $g_*: \mathrm{Hom}(X \otimes Y, Z) \to \mathrm{Hom}(X \otimes Y, W)$, and $g_*: \mathrm{Bil}(X \times Y, Z) \to \mathrm{Bil}(X \times Y, W)$. But if $f: X \otimes Y \to Z$, this says only that $(g_* \circ \pi^*)(f) = g_*(f \circ \pi) = g \circ f \circ \pi = \pi^*(g \circ f) = \pi^*(g_*(f)) = (\pi^* \circ g_*)(f)$, i.e. as usual this is merely the associativity of composition. **QED.**

Exercise #176) (i) Prove, for $x, \tilde{x}$, in X, $y, \tilde{y}$ in Y, and r in R, that $(x+\tilde{x}) \otimes y = x \otimes y + \tilde{x} \otimes y$, $x \otimes (y+\tilde{y}) = x \otimes y + x \otimes \tilde{y}$, and $r(x \otimes y) = (rx) \otimes y = x \otimes (ry)$ in $X \otimes_R Y$.
(ii) If $f: M \to N$ is linear, and y any element of Y, prove the map

$\tilde{f}:M\to N\otimes Y$ where $\tilde{f}(m) = f(m)\otimes y$, is linear.

Properties of the Tensor Product Functor

The main problem facing us with the tensor product is computing it. I.e. even after studying the definition, most people will have no clue what the module $X\otimes_R Y$ is, although they may think they know quite well what X and Y are. For instance, what is $\mathbb{R}\otimes_{\mathbb{Z}}\mathbb{Q}$? or even $\mathbb{Z}_n\otimes_{\mathbb{Z}}\mathbb{Z}_m$? This is to be expected since $X\otimes Y$ is a quotient of a gigantic free module by a large submodule of very complicated definition. It is not even obvious when $x\otimes y$ is zero for given elements x,y of X and Y, much less when $\Sigma_i x_i\otimes y_i = 0$. In some sense this mystery never entirely goes away, but wonderful strides can be made in calculating with tensor products by establishing a few simple properties. The situation is a little like that presented by the derivative in beginning calculus: the definition is a bit cumbersome but a few rules make computation of concrete examples very practical. [Maybe an analogy with computing antiderivatives is more accurate.] So we begin by asking how tensor products act on the fundamental constructions for forming modules, such as direct sums, submodules, and quotient modules. It turns out tensor products commute with direct sums, and with forming cokernels, but not with forming kernels. Thus in some sense they preserve quotient objects but not subobjects, and even the "preservation of quotient objects" must be understood properly. Moreover $R\otimes_R(\cdot)$ is equivalent to the identity functor, so combining this with the direct sum property, one can at least calculate the result of tensoring with direct sums of cyclic modules, such as free modules.

We usually fix one variable in the tensor product to get a functor in the other variable. Eg. for each module Y, the functor $(\cdot)\otimes Y:\mathfrak{M}\to\mathfrak{M}$, takes X to $X\otimes Y$, and takes $f:X\to\tilde{X}$ to $(f\otimes 1):X\otimes Y\to\tilde{X}\otimes Y$, (cf. Ex. 164 below). We want to derive some of its basic properties: eg. right exactness and commutativity with direct sums. Along the way we will prove other fundamental results, including "commutativity" and "associativity", and derive consequences such as the commutativity of tensor products with homomorphisms of free modules.

One can also consider $\otimes$ as a functor of two variables.

Exercise #177) If $f:M\to X$ and $g:N\to Y$ are linear maps, prove there is a unique linear map $f\otimes g:M\otimes N\to X\otimes Y$ such that $(f\otimes g)(m\otimes n) =$

$f(m)\otimes g(n)$, for all m in M, n in N. [In particular, this holds if $g = 1_Y$.]

Tensor product commutes with direct sums of modules
Lemma: For every module Y, the functor $F(\cdot) = (\cdot)\otimes Y$ commutes with direct sums. In particular, $(\oplus X_\alpha)\otimes Y \cong \oplus(X_\alpha \otimes Y)$ for any family $\{X_\alpha\}$.
proof: We give a fancy proof of this, why not? If $\{X_\alpha\}$ is a family of modules and Y a module, we have described equivalences of functors $Hom((\oplus X_\alpha)\otimes Y,\cdot) \cong Hom(\oplus X_\alpha, Hom(Y,\cdot)) \cong \prod Hom(X_\alpha, Hom(Y,\cdot)) \cong \prod Hom(X_\alpha \otimes Y,\cdot) \cong Hom(\oplus(X_\alpha \otimes Y),\cdot)$. Since $(\oplus X_\alpha)\otimes Y$ and $\oplus(X_\alpha \otimes Y)$ represent equivalent functors, they are isomorphic! QED.

Remark: Chasing through the equivalences above shows the isomorphism to be the map taking $\{x_\alpha\}\otimes y$ to $\{x_\alpha \otimes y\}$, as expected.

Exercise #178) Give the usual proof of the previous lemma; i.e. show the function $\tilde{\Theta}: (\oplus X_\alpha)\times Y \to \oplus(X_\alpha \otimes Y)$ taking $(\{x_\alpha\},y)$ to $\{x_\alpha \otimes y\}$ is bilinear, and induces an isomorphism $\Theta: (\oplus X_\alpha)\otimes Y \to \oplus(X_\alpha \otimes Y)$.

As usual, it is even more important to know how tensor products act on maps.
Tensor product commutes with direct sums of maps.
Lemma: Let $\{f_\alpha: X_\alpha \to W\}$, and $g: Y \to Z$, be module maps and $\Theta: (\oplus_\alpha X_\alpha)\otimes Y \to \oplus_\alpha(X_\alpha \otimes Y)$ the isomorphism in Ex. 165. Then $\Theta^*(\oplus(f_\alpha \otimes g)) = (\oplus f_\alpha)\otimes g$.
proof: The map $(\oplus f_\alpha)\otimes g: (\oplus_\alpha X_\alpha)\otimes Y \to W\otimes Z$, takes $\{x_\alpha\}\otimes y$ to $(\Sigma f_\alpha(x_\alpha))\otimes g(y)$, while the map $\Theta^*(\oplus(f_\alpha \otimes g))$ takes $\{x_\alpha\}\otimes y$ to $\oplus(f_\alpha \otimes g)(\Theta(\{x_\alpha\}\otimes y)) = \oplus(f_\alpha \otimes g)(\{x_\alpha \otimes y\})\ \Sigma(f_\alpha(x_\alpha)\otimes g(y))$. Since in $W\otimes Z$, $(\Sigma f_\alpha(x_\alpha))\otimes g(y) = \Sigma(f_\alpha(x_\alpha)\otimes g(y))$, we see our two maps agree on the usual generators of $(\oplus_\alpha X_\alpha)\otimes Y$, hence agree everywhere. QED.

Tensoring with R is equivalent to the identity functor
Lemma: For any R-module M, $R\otimes_R M \cong M$.
proof: The equivalences $Hom(R\otimes M,\cdot) \cong Hom(R,Hom(M,\cdot)) \cong Hom(M,\cdot)$, yield an isomorphism $R\otimes M \cong M$. QED.

Remarks: (i) Chasing the equivalences shows the isomorphism is the map taking $r\otimes m$ to rm. The usual proof is to show the function

$R\times M\to M$, defined by $(r,m)\mapsto rm$, is bilinear hence induces a linear map $\Theta_M:R\otimes M\to M$, with inverse the map $M\to R\otimes M$, $m\mapsto 1\otimes m$.
(ii) The isomorphism Θ is natural in M, i.e. given $f:M\to N$, the maps $(\Theta_N\circ(1\otimes f))$ and $(f\circ\Theta_M)$ are equal from $R\otimes M\to N$, since $(\Theta_N\circ(1\otimes f))(r\otimes m) = rf(m) = f(rm) = f(\Theta_M(r\otimes m)) = (f\circ\Theta_M)(r\otimes m)$.
Hence $R\otimes(\cdot)$, like $Hom(R,\cdot)$, is equivalent to the identity functor.

Corollary: Tensoring with the free module $\oplus_A R$ is equivalent to the direct sum functor $\oplus_A$; i.e. the natural map $\Theta:(\oplus_\alpha R)\otimes X\to\oplus_\alpha X$, taking $(r_\alpha)\otimes x$ to $(r_\alpha x)$ is an isomorphism, and given $f:X\to Y$, the map $(1\otimes f):(\oplus_\alpha R)\otimes X\to(\oplus_\alpha R)\otimes Y$ corresponds to $(\oplus_\alpha f):\oplus_\alpha X\to\oplus_\alpha Y$, under the isomorphisms Θ_X and Θ_Y; i.e. $(\Theta_Y)\circ(1\otimes f) = (\Theta_X)\circ(\oplus_\alpha f)$.

Examples: $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}^m \cong \mathbb{Q}^m$; $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^m \cong \mathbb{C}^m$.

Application to invariance of the rank of free modules
Tensor products now render "mechanical" an argument we have used before:
Corollary: The rank of a free module is well defined over any ring, even when the rank is infinite.
proof: We reduce to the case of a vector space, (for which see appendix). Let $f:X=\oplus_A R\to\oplus_B R=Y$ be an R-isomorphism, and let $I\subset R$ be a maximal ideal. Then f induces an R-isomorphism $(f\otimes 1)\ \ X\otimes(R/I)\to Y\otimes(R/I)$, hence $\oplus_A(R/I)\cong\oplus_A(R\otimes R/I)\cong(\oplus_A R)\otimes(R/I) = X\otimes(R/I)\cong Y\otimes(R/I)\cong\oplus_B(R/I)$. Then, since $\oplus_A(R/I)\cong\oplus_B(R/I)$ is an R-isomorphism of modules whose annihilators contain I, it is also an R/I vector space isomorphism. Thus $A\approx B$ by invariance of dimension of vector spaces. QED.

"Commutativity" of tensor products:
Lemma: For all X,Y we have $X\otimes Y\cong Y\otimes X$
proof: The following functors are equivalent: $Hom(X\otimes Y,\cdot)\cong Bil(X\times Y,\cdot)\cong Bil(Y\times X,\cdot)\cong Hom(Y\otimes X,\cdot)$. QED.

Remarks: (i) The equivalence $Bil(X\times Y,\cdot)\cong Bil(Y\times X,\cdot)$ induced by interchanging variables induces the map $X\otimes Y\to Y\otimes X$, which thus takes $x\otimes y$ to $y\otimes x$, as expected.
(ii) For maps $f:X\to Z$, $g:Y\to W$, $(f\otimes g): X\otimes Y\to Z\otimes W$ corresponds to $(g\otimes f): Y\otimes X\to W\otimes Z$ under these isomorphisms.

Corollary: Tensor product commutes with direct sums in both variables, i.e. $(\oplus_A X_\alpha)\otimes(\oplus_B Y_\beta) \cong \oplus_A\oplus_B(X_\alpha\otimes Y_\beta) \cong \oplus_{A\times B}(X_\alpha\times Y_\beta)$.

Remark: For maps, given $\{f_\alpha\ X_\alpha \to W_\alpha\}$, and $\{g_\beta:Y_\beta \to Z_\beta\}$, $\oplus_{\alpha,\beta}(f_\alpha\otimes g_\beta)$ corresponds to $(\oplus_\alpha f_\alpha)\otimes(\oplus_\beta g_\beta)$ under the isomorphisms in the corollary.

Application: Computation of tensor products of free modules.
Corollary: If $X = \oplus_A R$, and $Y = \oplus_B R$ are free modules on the sets A,B, then $(\oplus_A R)\otimes(\oplus_B R) \cong \oplus_{A\times B} R$ is free on the set $A\times B$. In particular, the rank of $X\otimes Y$ is the product of the ranks of X and Y.

Remarks:(i) The maps $(\oplus_A R)\otimes(\oplus_B R)\to \oplus_A(R\otimes(\oplus_B R))\to \oplus_{A\times B}(R\otimes R)$ take $\{r_\alpha\}\otimes\{r_\beta\}\mapsto\{r_\alpha\otimes\{r_\beta\}\}\mapsto\{r_\alpha\otimes r_\beta\}$; hence the isomorphism $(\oplus_A R)\otimes(\oplus_B R)\to \oplus_{A\times B}R$ takes $\{r_\alpha\}\otimes\{r_\beta\}\mapsto\{r_\alpha r_\beta\}$. I.e. $\{r_\alpha\}\otimes\{r_\beta\}$ goes to $\{r(\alpha,\beta)\}$ where $r(\alpha,\beta) = r_\alpha r_\beta$. In particular if e_α, e_β are standard basis elements of $(\oplus_A R)$ and $(\oplus_B R)$, then under the isomorphism above the standard basis element $e_{\alpha,\beta}$ of $\oplus_{A\times B}R$ corresponds to $e_\alpha\otimes e_\beta$ in $(\oplus_A R)\otimes(\oplus_B R)$.
(ii) In particular, if $S = \{x_i\}$ is a basis of the free module X, and $T = \{y_j\}$ is a basis of the free module Y, then the set $C = \{x_i\otimes y_j$, for x_i in S, y_j in T$\}$ is a basis of the free module $X\otimes Y$.
(iii) Given maps $f_\alpha:R\to M$ taking $1\mapsto x_\alpha$, and $g_\beta:R\to Y$ taking $1\mapsto y_\beta$, we get maps $\oplus f_\alpha:\oplus_\alpha R\to M$ taking taking $e_\alpha\mapsto x_\alpha$,and $\oplus g_\beta:\oplus_\beta R\to Y$ taking $e_\beta\mapsto y_\beta$. Applying $\otimes$, $(\oplus f_\alpha)\otimes(\oplus g_\beta):(\oplus_A R)\otimes(\oplus_B R)\to M\otimes Y$ takes $e_\alpha\otimes e_\beta\mapsto x_\alpha\otimes y_\beta$, and in the other order $\oplus(f_\alpha\otimes g_\beta):\oplus_{A\times B}(R\otimes R) \cong \oplus_{A\times B}(R)\to M\otimes Y$ also takes $e_{\alpha,\beta}\mapsto x_\alpha\otimes y_\beta$. Thus $(\oplus f_\alpha)\otimes(\oplus g_\beta)$ corresponds to $\oplus(f_\alpha\otimes g_\beta)$, as claimed above.

Next, one more desirable property for a "product" to have.
Associativity of tensor products
Lemma: For all X,Y,Z, we have $X\otimes(Y\otimes Z) \cong (X\otimes Y)\otimes Z$.
proof: These functors are equivalent: $\mathrm{Hom}(X\otimes(Y\otimes Z),\cdot) \cong \mathrm{Hom}(X,\mathrm{Hom}(Y\otimes Z,\cdot)) \cong \mathrm{Hom}(X,\mathrm{Hom}(Y,\mathrm{Hom}(Z,\cdot))) \cong \mathrm{Hom}(X\otimes Y,\mathrm{Hom}(Z,\cdot)) \cong \mathrm{Hom}((X\otimes Y)\otimes Z,\cdot)$. QED.

Remarks: (i) Tracing the equivalences shows $x\otimes(y\otimes z)\mapsto(x\otimes y)\otimes z$.
(ii) For maps $f:X\to M$, $g:Y\to N$, $h:Z\to P$, the induced maps

$f\otimes(g\otimes h): X\otimes(Y\otimes Z)\to M\otimes(N\otimes P)$ and $(f\otimes g)\otimes h:(X\otimes Y)\otimes Z\to(M\otimes N)\otimes P$ also correspond via the associativity isomorphisms above.
(iii) By (ii) the isomorphisms in the lemma are natural in X,Y,Z, but the same is true of some other more eccentric isomorphisms, such as the maps $X\otimes(Y\otimes Z)\to(X\otimes Y)\otimes Z$ taking $x\otimes(y\otimes z)$ to $-(x\otimes y)\otimes z$. The problem with this latter choice of "natural" isomorphisms is that if we compose them in the following sequence: $(X\otimes Y)\otimes(Z\otimes W)\cong ((X\otimes Y)\otimes Z)\otimes W \cong (X\otimes(Y\otimes Z))\otimes W \cong X\otimes((Y\otimes Z)\otimes W) \cong X\otimes(Y\otimes(Z\otimes W)) \cong (X\otimes Y)\otimes(Z\otimes W)$, the composite $(X\otimes Y)\otimes(Z\otimes W)\cong(X\otimes Y)\otimes(Z\otimes W)$ is not the identity, but -id. This is an example of a family of isomorphisms which are not "coherent", even though they are "natural". Of course only a very eccentric person would choose the isomorphisms this way, but sometimes it might not be so obvious which isomorphisms will turn out to be "coherent". [Dave Benson kindly pointed out this interesting subtlety. We might perhaps define a family of maps among a family objects to be coherent iff there is one map given for each ordered pair of objects, the map of each object to itself is the identity, and the family of maps is closed under composition.]

Modules of n-multilinear Maps

Another approach to associativity is to notice that both $X\otimes(Y\otimes Z)$ and $(X\otimes Y)\otimes Z$ represent the functor $\mathrm{Hom}(X,\mathrm{Hom}(Y,\mathrm{Hom}(Z,\cdot)))\cong$ {trilinear maps on $X\times Y\times Z$}. Hence the maps $(x,y,z)\mapsto x\otimes(y\otimes z)$ and $(x,y,z)\mapsto(x\otimes y)\otimes z$ must induce natural isomorphisms between them. It is useful to have the concept of higher multilinear maps available, since some very important examples of them exist, such as the triple product $v\cdot(w\times u)$ from vector calculus, the determinant, and the curvature tensor in differential geometry. Because the constructions mirror those for bilinear maps, we only sketch them.

Definition: (i) Given modules $X_1,\ldots,X_n,Y$, a function $f:X_1\times\ldots\times X_n\to Y$ is n-multilinear, or simply n-linear, iff it is linear in each variable separately. For instance, $f(v_1,\ldots,v_n)$ is linear in the first variable, iff for all choices of $v_2,\ldots,v_n$, the function $X_1\to R$ defined by $v\mapsto f(v,v_2,\ldots,v_n)$, is linear
(ii) NOTE: This does NOT say that $f(v,v_2,\ldots,v_n)+f(\tilde{v},v_2,\ldots,v_n) = f((v,v_2,\ldots,v_n)+(\tilde{v},v_2,\ldots,v_n))$!! Rather it implies that $f(v,v_2,\ldots,v_n)+f(\tilde{v},v_2,\ldots,v_n) = f(v+\tilde{v},v_2,\ldots,v_n)$. Similarly it does <u>not</u> say that $r\cdot f(v,v_2,\ldots,v_n) = f(r\cdot(v,v_2,\ldots,v_n))$; rather it implies that

$r\cdot f(v,v_2,\ldots,v_n) = f(r\cdot v,v_2,\ldots,v_n)$. I.e. the addition and multiplication which occur in the "source", are not those in the Cartesian product, but rather the operations in X_1.
(iii) We denote the set of n-linear maps $X_1\times\ldots\times X_n\to Y$ by $\mathcal{L}^n(X_1\times\ldots\times X_n;Y)$, by $\mathcal{L}^n(X;Y)$ if all $X_i=X$, and by $\mathcal{L}^n(X)$ if also $Y=R$. In every case, $\mathcal{L}^n(X_1\times\ldots\times X_n;Y)$ is an R-module, with operations taken pointwise in Y.
(iv) If $f:Y\to Z$ is linear, and $g:X_1\times\ldots\times X_n\to Y$ is n-linear, the composition $f_*(g):X_1\times\ldots\times X_n\to Z$ is n-linear, and the resulting map $f_*:\mathcal{L}^n(X_1\times\ldots\times X_n;Y)\to\mathcal{L}^n(X_1\times\ldots\times X_n;Z)$ is linear. Since as usual, $(f\circ h)_* = f_*\circ h_*$, and $id_* = id$, $\mathcal{L}^n(X_1\times\ldots\times X_n;\cdot)$ is a functor $\mathfrak{M}\to\mathfrak{M}$.

Example: Under the isomorphism $Mat_n(R)\cong R^n\times\ldots\times R^n$, given by considering a matrix as the sequence of its rows, the determinant function $\det:Mat_n(R)\to R$, corresponds to an element of $\mathcal{L}^n(R^n)$.

Theorem: The functor $\mathcal{L}^n(X_1\times\ldots\times X_n;\cdot)$ is representable.
proof: We imitate the construction of the tensor product of two modules. I.e. begin with the free module F on the set $X_1\times\ldots\times X_n$. Then define the submodule $K\subset F$ generated by the multilinear relations and set $X_1\otimes\ldots\otimes X_n = F/K$. Thus $X_1\otimes\ldots\otimes X_n$ is the R-module generated by the symbols $x_1\otimes\ldots\otimes x_n$, for all choices of x_i in X_i, and satisfying the relations $(x_1\otimes\ldots\otimes[x_i+\tilde{x}_i]\otimes\ldots\otimes x_n) = (x_1\otimes\ldots\otimes x_i\otimes\ldots\otimes x_n) + (x_1\otimes\ldots\otimes\tilde{x}_i\otimes\ldots\otimes x_n)$, and $r(x_1\otimes\ldots\otimes x_i\otimes\ldots\otimes x_n) = (x_1\otimes\ldots\otimes rx_i\otimes\ldots\otimes x_n)$. Then we proceed as before:
(i) The function $\Theta:X_1\times\ldots\times X_n\to X_1\otimes\ldots\otimes X_n$, defined by $(x_1,\ldots,x_n)\mapsto(x_1\otimes\ldots\otimes x_n)$, is n-linear.
(ii) For every linear function $f:X_1\otimes\ldots\otimes X_n\to Y$, the composition $(f\circ\Theta):X_1\times\ldots\times X_n\to Y$, is n-linear.
(iii) Every n-linear function $X_1\times\ldots\times X_n\to Y$ induces a linear function $F\to Y$ which vanishes on $K\subset F$, hence induces a linear function $X_1\otimes\ldots\otimes X_n\to Y$.
(iv) Consequently, for every Y the n-linear function $\Theta:(X_1\times\ldots\times X_n)\to X_1\otimes\ldots\otimes X_n$, induces an isomorphism $\Theta^*:Hom(X_1\otimes\ldots\otimes X_n,Y)\to\mathcal{L}^n(X_1\times\ldots\times X_n;Y)$, which is natural in Y.
QED.

Exercise #179) In the previous theorem, if F is the free module on the set $X_1 \times .. \times X_n$, write down the generators for the submodule $K \subset F$ of "multilinear relations" such that $X_1 \otimes .... \otimes X_n = F/K$, and prove that the function $\otimes : X_1 \times .... \times X_n \to X_1 \otimes ... \otimes X_n$, defined by $(x_1, ..., x_n) \mapsto (x_1 \otimes .... \otimes x_n)$, is n-linear.

Corollary: Every way of associating the product $X_1 \otimes .... \otimes X_n$, such as $X_1 \otimes (X_2 \otimes .... \otimes X_n)$, is canonically isomorphic to $X_1 \otimes .... \otimes X_n$.
proof: They all represent the same functor $\mathcal{L}^n(X_1 \times ... \times X_n; \cdot)$. **QED.**

Corollary: Every way of permuting the factors $X_i \mapsto X_{\sigma(i)}$ yields an isomorphism $X_1 \otimes ... \otimes X_n \cong X_{\sigma(1)} \otimes .... \otimes X_{\sigma(n)}$.
proof: The functors $\mathcal{L}^n(X_1 \times ... \times X_n; \cdot)$ and $\mathcal{L}^n(X_{\sigma(1)} \times ... \times X_{\sigma(n)}; \cdot)$ are equivalent. **QED.**

Corollary: The tensor product $X_1 \otimes .... \otimes X_n$ commutes with direct sums in each variable separately.

Right Exactness of Tensor Products

The other fundamental characteristic of the tensor product, after commutativity with direct sums, is right exactness. Since tensor products are defined as objects representing composed Hom functors, it should not be too surprising that we intend to prove right exactness of tensor products by exploiting the left exactness of Hom. We will need to be clear about what happens to maps when we replace a tensor product by a repeated Hom functor. In particular we must verify that the equivalence $\text{Hom}(X \otimes A, M) \cong \text{Hom}(X, \text{Hom}(A, M))$, which we know is natural in the variable M, is natural also in the variable A. This is quite believable, but we prove it as an illustration of how to trace natural equivalences. The proof of the next lemma is a bit tedious, just writing down some entirely predictable maps, composing them carefully two ways, and comparing the results. At least it is part of the general theory of tensor products and not specific only to the following theorem.

Lemma: Composing the isomorphisms $\text{Hom}(X \otimes A, M) \cong \text{Bil}(X \times A, M) \cong \text{Hom}(X, \text{Hom}(A, M))$ described above, yields a natural equivalence of contravariant functors $\text{Hom}(X \otimes (\cdot), M) \cong \text{Hom}(X, \text{Hom}(\cdot, M))$.
proof: For each A, $\varphi_A : \text{Hom}(X \otimes A, M) \to \text{Hom}(X, \text{Hom}(A, M))$ takes the

map λ to $\varphi_A(\lambda) = \tilde{\lambda}$, where $\tilde{\lambda}(x)(a) = \lambda(x \otimes a)$. We know [from ex. 163 and the defining property of tensor product], that φ_A is an isomorphism. Let's recall the meaning of naturality in A.

On the one hand, given $f{:}A \to B$, the functor $X \otimes (\cdot)$ yields a unique map $(1 \otimes f){:}X \otimes A \to X \otimes B$ such that $(1 \otimes f)(x \otimes a) = x \otimes f(a)$. The functor $Hom(\cdot, M)$ applied to $(1 \otimes f)$ then induces the unique map $(1 \otimes f)^*{:}Hom(X \otimes B, M) \to Hom(X \otimes A, M)$, such that $(1 \otimes f)^*(\lambda) = \lambda \circ (1 \otimes f)$.

On the other hand, $Hom(\cdot, M)$ applied to f yields the map $f^*{:}Hom(B,M) \to Hom(A,M)$, where $f^*(\alpha) = \alpha \circ f$. Then applying $Hom(X, \cdot)$ to f^* yields $(f^*)_*{:}Hom(X, Hom(B,M)) \to Hom(X, Hom(A,M))$, where $(f^*)_*(\mu)(x) = (f^* \circ \mu)(x) = f^*(\mu(x)) = \mu(x) \circ f$.

The goal is to show $\varphi_A{:}Hom(X \otimes A, M) \to Hom(X, Hom(A,M))$, and $\varphi_B{:}Hom(X \otimes B, M) \to Hom(X, Hom(B,M))$, as given above, are compatible with the maps $(1 \otimes f)^*$ and $(f^*)_*$. I.e. we claim the diagram below commutes:

$$
\begin{array}{ccc}
Hom(X \otimes B, M) & \xrightarrow{(1 \otimes f)^*} & Hom(X \otimes A, M) \\
\downarrow \varphi_B & & \downarrow \varphi_A \\
Hom(X, Hom(B,M)) & \xrightarrow{(f^*)_*} & Hom(X, Hom(A,M))
\end{array}
$$

To this end let $\lambda{:}X \otimes B \to M$, and consider $[((\varphi_A \circ (1 \otimes f)^*)(\mu))(x)](a) = [(\varphi_A)(\mu \circ (1 \otimes f))(x)](a) = \mu(x \otimes f(a))$. In the other direction, we have $[(((f^*)_* \circ \varphi_B)(\mu))(x)](a) = [\varphi_B(\mu)(x) \circ f](a) = \varphi_B(\mu)(x)(f(a)) = \mu(x \otimes f(a))$. Since the results are the same, we are done. QED.

Now we get right exactness of $X \otimes (\cdot)$ easily.

Theorem: If $A \to B \to C \to 0$ is an exact sequence, and X any module, then the sequence $X \otimes A \to X \otimes B \to X \otimes C \to 0$ is exact also.

proof: By the converse of Ex 160, proved above, it suffices to show for all modules M, that the sequence

$0 \to Hom(X \otimes C, M) \to Hom(X \otimes B, M) \to Hom(X \otimes A, M)$ is exact. By the previous lemma and the next exercise, this last sequence is exact iff the following sequence is exact:
$0 \to Hom(X, Hom(C,M)) \to Hom(X, Hom(B,M)) \to Hom(X, Hom(A,M))$.
Since $Hom(\cdot, M)$ changes right exact sequences into left exact ones, and $Hom(X, \cdot)$ preserves left exactness, this last sequence is exact.

Exercise #180) If, in the following commutative diagram, the vertical maps are isomorphisms and the top row is exact, then the bottom row is also exact:

$$\begin{array}{ccccccc} 0 & \to & A & \to & B & \to & C \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & X & \to & Y & \to & Z \end{array}$$

QED.

Corollary: For any X, the functor $X \otimes (\cdot)$ is right exact.

One more result is very useful for computations.
Corollary: For all R-modules M, and all ideals $\alpha \subset R$, there is an isomorphism $M/\alpha M \cong M \otimes_R (R/\alpha)$, taking $[x] \mapsto x \otimes [1]$.
proof: Consider the exact sequence $0 \to \alpha \to R \to R/\alpha \to 0$, and tensor it with M, getting the exact sequence $M \otimes \alpha \to M \otimes R \to M \otimes (R/\alpha) \to 0$. Thus $M \otimes (R/\alpha) \cong (M \otimes R)/Im(M \otimes \alpha)$. Since $M \otimes R \cong M$ by the map taking $(m \otimes r) \mapsto rm$, this implies $M \otimes (R/\alpha) \cong M/Im(M \otimes \alpha \to M) = M/\alpha M$. The isomorphism $M/\alpha M \to M \otimes (R/\alpha)$ thus is induced by the map $M \to M \otimes R \to M \otimes R/\alpha$ taking x to $x \otimes [1]$. QED.

Exercise #181) Give an elementary direct proof of the previous corollary as follows. Define the linear map $\tilde{f}: M \to M \otimes R/\alpha$ by $\tilde{f}(x) = x \otimes [1]$, and prove there is a unique induced map $f: M/\alpha M \to M \otimes R/\alpha$. In the other direction, define an appropriate (obvious) bilinear function $\tilde{g}: M \times R/\alpha \to M/\alpha M$, which induces a linear map $g: M \otimes R/\alpha \to M/\alpha M$, which is inverse to f.

Corollary: If d = gcd(n,m) then $\mathbf{Z}_n \otimes_{\mathbf{Z}} \mathbf{Z}_m \cong \mathbf{Z}_d$.

Examples: $\mathbf{Z}_{125} \otimes_{\mathbf{Z}} \mathbf{Z}_{16} \cong \{0\}$; $(\mathbf{Z}^2 \times \mathbf{Z}_{12}) \otimes (\mathbf{Z}^3 \times \mathbf{Z}_2) \cong \mathbf{Z}^6 \times (\mathbf{Z}_2)^3 \times (\mathbf{Z}_{12})^3$.

Remark: In particular if n,m are relatively prime, and M is any abelian group, then the only bilinear function $f: \mathbf{Z}_n \times \mathbf{Z}_m \to M$ is $f=0$.

Corollary: One can explicitly compute the tensor product, over **Z**, of any two direct sums of cyclic **Z**-modules, hence in principle of any two finitely generated abelian groups

Exercise #182) Prove the results in the previous two corollaries.

Question: Do the results of the previous two corollaries generalize to other rings?

Remarks: (i) Exercise 169 shows the computational results in the previous three corollaries could have been proved earlier, in particular before right exactness, which would have made them seem more elementary. I chose instead to emphasize the fundamental importance for tensor products of right exactness and commutativity with direct sums, by showing how the other properties flow naturally from those two.
(ii) In fact if $F(\cdot)$ is any right exact, linear functor, that commutes with direct sums, then $F(\cdot) \cong F(R) \otimes (\cdot)$ [Eilenberg-Watts,1960]. To sketch how the argument goes, let M be any module and represent M as a cokernel of a map of free modules as before: $\oplus_K R \to \oplus_M R \to M \to 0$ Apply first $F(\cdot)$, then $F(R)\otimes(\cdot)$, to get sequences equivalent to the following ones: $\oplus_K F(R) \to \oplus_M F(R) \to F(M) \to 0$, and $\oplus_K F(R) \to \oplus_M F(R) \to F(R)\otimes M \to 0$. Then show the two maps at the left ends of these sequences are the same, and apply uniqueness of cokernels to get $F(M) \cong F(R)\otimes M$. Finally check naturality in M.

Tensor Products of Homomorphisms

The natural map which tensors two linear maps together to get one linear map on the tensor product of the domains, is an isomorphism when all modules involved are finite and free, and it yields several standard isomorphisms that occur frequently in the literature. Two of the most common special cases (actually equivalent) are the isomorphisms $M^* \otimes N \cong \mathrm{Hom}(M,N)$, and $M^* \otimes N^* \cong (M\otimes N)^*$, valid in particular for finite dimensional vector spaces M,N. Since the module of homomorphisms of two finite free modules is also free, this discussion is a special case of the fact that the tensor product of two free modules is free From that point of view the following lemma is a corollary of results already proved, but it seems necessary for clarity to spell out again the explicit maps which occur in the present situation.

In general, let A,B,C,D be modules, let f be in Hom(A,B) and g in Hom(C,D). Consider the function $A \times C \to B \otimes D$ which takes (a,c) to $f(a) \otimes g(c)$. This is immediately seen to be bilinear, hence induces a linear map $T(f,g): A \otimes C \to B \otimes D$, such that $T(f,g)(a \otimes c) = f(a) \otimes g(c)$. Next consider the function $T: Hom(A,B) \times Hom(C,D) \to Hom(A \otimes C, B \otimes D)$, which takes (f,g) to the map T(f,g). Then T is bilinear as well, hence induces the linear map $\tilde{T}: Hom(A,B) \otimes Hom(C,D) \to Hom(A \otimes C, B \otimes D)$ such that $\tilde{T}(f \otimes g) = T(f,g)$. [Note the possibility of confusion in notation, since it is plausible to denote T(f,g) also by $f \otimes g$, as we did in Ex.164. Of course since $f \otimes g$ and T(f,g) correspond under a natural map, it may seem less important to distinguish them, at least in the case where the modules are finite and free.]

Lemma: The map $\tilde{T}: Hom(A,B) \otimes Hom(C,D) \to Hom(A \otimes C, B \otimes D)$ such that $\tilde{T}(f \otimes g) = T(f,g)$, is an isomorphism, when A,B,C,D are all finite rank free modules.

proof: It is simple to check that a basis goes to a basis as follows: let $\{\alpha_j\}$, $\{\beta_i\}$, $\{\gamma_t\}$, $\{\delta_s\}$ be bases of A,B,C,D respectively, and define $\{\varphi_{ij}\}$ in Hom(A,B), $\{\psi_{st}\}$ in Hom(C,D), by setting $\varphi_{ij}(\alpha_j) = \beta_i$, $\psi_{st}(\gamma_t) = \delta_s$, and $\varphi_{ij}(\alpha_k) = 0 = \psi_{st}(\gamma_u)$ if $k \neq j$, $u \neq t$. Then $\{\varphi_{ij}\}$, $\{\psi_{st}\}$ are bases of Hom(A,B) and Hom(C,D) respectively. (One way to see this is to use the bases of A,B, say, to construct an isomorphism between Hom(A,B) and matrices of an appropriate size. Then the linear map φ_{ij} has the matrix with (i,j) entry 1, and all others 0, hence $\{\varphi_{ij}\}$ form a basis of all matrices.) We know too from our discussion of tensor products of free modules that $\{\alpha_j \otimes \gamma_t\}$, and $\{\beta_i \otimes \delta_s\}$ are bases of $A \otimes C$ and $B \otimes D$ respectively. Consequently a basis of $Hom(A \otimes C, B \otimes D)$ is given by $\{\eta_{isjt}\}$ where $\eta_{isjt}(\alpha_j \otimes \gamma_t) = \beta_i \otimes \delta_s$, and η_{isjt} vanishes on every other basis element. Similarly, $\{\varphi_{ij} \otimes \psi_{st}\}$ is a basis of $Hom(A,B) \otimes Hom(C,D)$. Now $\tilde{T}(\varphi_{ij} \otimes \psi_{st})(\alpha_k \otimes \gamma_u) = \varphi_{ij}(\alpha_k) \otimes \psi_{st}(\gamma_u) = 0$ unless $k = j$, $u = t$, when it equals $\beta_i \otimes \delta_s$. Since $\tilde{T}$ takes the basis $\{\alpha_j \otimes \gamma_t\}$ to the basis $\{\eta_{isjt}\}$, it is an isomorphism. QED.

Remarks: (i) This result gives a tensor product operation on matrices. Eg if the free modules A,B,C,D have ranks m, n, p, q, respectively, then f in Hom(A,B), g in Hom(C,D) may be considered matrices of dimensions $n\times m$, $q\times p$ respectively, and $\tilde{T}(f\otimes g)$ an $nq\times mp$ matrix. The rows of $[\tilde{T}(f\otimes g)]$ are indexed by the pairs (i,s) and the columns by the pairs (j,t), which must be ordered in some way to yield a matrix, but however that is chosen, if the (i,j) entry of f is x_{ij} and the (s,t) entry of g is y_{st}, then the ((i,s),(j,t)) entry of $[\tilde{T}(f\otimes g)]$ is the product $x_{ij}y_{st}$. For example, if we use the lexicographical ordering on pairs, then we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} ax & ay & bx & by \\ az & aw & bz & bw \\ cx & cy & dx & dy \\ cz & cw & dz & dw \end{bmatrix}$$

, (where, in the notation of the theorem, the right side would be denoted as $\tilde{T}$ of the left side). For example, the third row on the right is, in the lexicographical ordering, the (2,1) row. Hence the entries all come from the 2nd row of the first factor matrix and the 1st row of the second factor matrix. The third entry in the third row, dx, is in the (2,1) column, hence its factors come from columns 2 and 1, respectively, of the factor matrices on the left side of the equation.

(ii) The complexity of this sort of calculation may be responsible for the fearsome reputation which "tensor analysis" once enjoyed. In ancient times, books on the topic were filled with lengthy formulas laden with indices. Learning the subject meant memorizing rules for manipulating those indices. Nowadays, confronted with the statement that such and such quantity is "a tensor", I hope we will understand this to mean simply the quantity has certain linearity properties with respect to each of its components. Of course skill in their use will still require an ability to calculate. In this regard, note that we are usually able to recover explicit calculations from our abstract approach, *provided we always know exactly what the maps are* that yield our isomorphisms. When we know the maps, a choice of bases gives us a calculation. Thus we must resist the tendency to remember only that certain modules are isomorphic, without knowing what the isomorphisms are. Fortunately the maps are virtually always the simplest ones we can think of.

Corollary: If M,N are finite free modules, then $M^*\otimes N \cong \mathrm{Hom}(M,N)$.

proof: $M^* = \mathrm{Hom}(M,R)$, and $N \cong \mathrm{Hom}(R,N)$ so the lemma gives $M^* \otimes N \cong \mathrm{Hom}(M,R) \otimes \mathrm{Hom}(R,N) \cong \mathrm{Hom}(M \otimes R, R \otimes N) \cong \mathrm{Hom}(M,N)$
Tracing the isomorphism shows that the map $M^* \otimes N \to \mathrm{Hom}(M,N)$ takes $\lambda \otimes n$ to the map sending m to $\lambda(m)n$. **QED.**

Corollary: If M,N are finite free modules, then $M^* \otimes N^* \cong (M \otimes N)^*$.
proof: Using the lemma, $M^* \otimes N^* = \mathrm{Hom}(M,R) \otimes \mathrm{Hom}(N,R) \cong \mathrm{Hom}(M \otimes N, R \otimes R) \cong \mathrm{Hom}(M \otimes N, R) = (M \otimes N)^*$ Tracing the isomorphisms shows that $\lambda \otimes \mu$ in $M^* \otimes N^*$ goes to the map taking $(m \otimes n)$ to $\lambda(m)\mu(n)$ QED.

Exercise #183) Assume M,N are finite free modules.
(i) Verify directly that the map $M^* \otimes N \to \mathrm{Hom}(M,N)$ taking $\lambda \otimes n$ to the homomorphism sending m to $\lambda(m)n$ is well defined and an isomorphism.
(ii) Do the same for the map $M^* \otimes N^* \to (M \otimes N)^*$ taking $\lambda \otimes \mu$ to the functional whose value on $(m \otimes n)$ is $\lambda(m)\mu(n)$.

A Word About "Flatness"
It is very useful to know when a module M has the property that $M \otimes (\cdot)$ is left exact as well as right exact

Definition: An R-module M is called "flat" (or R-flat), iff the functor $M \otimes (\cdot)$ is exact; i.e. iff whenever $0 \to A \to B \to C \to 0$ is exact, then the induced sequence $0 \to M \otimes A \to M \otimes B \to M \otimes C \to 0$ is also exact.

Remark: (i) It suffices for flatness of M, to check that whenever $0 \to A \to B$ is exact, then $0 \to M \otimes A \to M \otimes B$ is also exact.
(ii) In algebraic geometry, maps corresponding to flat algebras have the property that all the fibers have the same algebraic invariants, same dimension, etc... Perhaps the term "flat" is a reference to the appearance of such a map, viewed as a fibering of the source space over the target space, since the fiber dimension never "jumps up" at special points.

Lemma: Every free module is flat.
Proof: We may assume $M = \oplus_A R$, since an isomorphism $M \cong \oplus_A R$ induces an equivalence of functors $M \otimes (\cdot) \cong (\oplus_A R) \otimes (\cdot)$. But we already know that $(\oplus_A R) \otimes (\cdot)$ is equivalent to the direct sum

functor, and a direct sum of injections is injective. I.e. Assume $f:X \to Y$ is injective; then $(1\otimes f):(\oplus_\alpha R)\otimes X \to (\oplus_\alpha R)\otimes Y$ is injective iff $(\oplus_\alpha f):(\oplus_\alpha X) \to (\oplus_\alpha Y)$ is injective. But for the latter map, $(\oplus_\alpha f)(\{x_\alpha\}) = \{f(x_\alpha)\} = \{0_\alpha\}$, iff $f(x_\alpha)=0$ for all α, iff $x_\alpha=0$, all α, since f is injective. Thus $(\oplus_\alpha f)$ and hence $(1\otimes f)$, are injective. QED.

In fact **Q** is a flat **Z**-module, but we prove only a partial result:

Lemma: If $R = \mathbf{Z}$, B is a finitely generated **Z**-module, and $0 \to A \to B$ is exact, then $0 \to \mathbf{Q}\otimes A \to \mathbf{Q}\otimes B$ is also exact.

proof: Let $f:A \to B$ be injective, and consider $(1\otimes f):\mathbf{Q}\otimes A \to \mathbf{Q}\otimes B$. We must show if $(1\otimes f)(\Sigma_i (x_i/y_i)\otimes a_i) = 0$, then $(\Sigma_i (x_i/y_i)\otimes a_i) = 0$. First choose a common denominator y for the fractions x_i/y_i so that we may assume all $y_i = y$. Then we have $(\Sigma_i (x_i/y_i)\otimes a_i) = (\Sigma_i(\tilde{x}_i/y)\otimes a_i) = (\Sigma_i (1/y)\otimes \tilde{x}_i a_i) = (1/y)\otimes(\Sigma_i \tilde{x}_i a_i)$. Thus $(1\otimes f)(\Sigma_i (x_i/y_i)\otimes a_i) = (1\otimes f)((1/y)\otimes(\Sigma_i \tilde{x}_i a_i)) = (1/y)\otimes(\Sigma_i \tilde{x}_i f(a_i)) = 0$

Claim: If B is a finitely generated **Z**-module, and if $(1/y)\otimes b = 0$ in $\mathbf{Q}\otimes B$, then b is a torsion element of B.

Assuming the claim, we conclude that $\Sigma_i \tilde{x}_i f(a_i) = f(\Sigma_i \tilde{x}_i a_i)$ is a torsion element of B, whence $\Sigma_i \tilde{x}_i a_i$ is torsion in A. But if $r(\Sigma_i \tilde{x}_i a_i) = 0$, for $r \neq 0$, then $0 = (1/ry)\otimes r(\Sigma_i \tilde{x}_i a_i) = (r/ry)\otimes(\Sigma_i \tilde{x}_i a_i) = (1/y)\otimes(\Sigma_i \tilde{x}_i a_i) = (\Sigma_i (x_i/y_i)\otimes a_i)$, proving the lemma.

To prove the claim, we know $B \cong \mathbf{Z}^n\oplus T$ where T is torsion, so b in B has form (x,y) for x in $\mathbf{Z}^n$, y in T. Then since $1\otimes b$ in $\mathbf{Q}\otimes B$ corresponds to $(1\otimes x, 1\otimes y)$ in $(\mathbf{Q}\otimes\mathbf{Z}^n) \oplus (\mathbf{Q}\otimes T)$, $1\otimes b = 0$ iff both $1\otimes x$ and $1\otimes y = 0$. But since $\mathbf{Z}^n$ is flat, $\mathbf{Z}^n \cong \mathbf{Z}\otimes\mathbf{Z}^n \to \mathbf{Q}\otimes\mathbf{Z}^n$ is injective, so $1\otimes x$ is zero in $\mathbf{Q}\otimes\mathbf{Z}^n$ iff x is zero in $\mathbf{Z}^n$. Hence from $1\otimes b = 0$, for $b = (x,y)$ in $\mathbf{Z}^n\oplus T$, we conclude $x=0$, hence b is torsion, proving the claim and the lemma. QED.

Even this partial result has useful consequences.

Lemma: For a fin gen **Z**-module $M \cong \mathbf{Z}^n\oplus T$, where T is torsion, $\mathbf{Q}\otimes M \cong \mathbf{Q}^n$ is a **Q**-vector space of dimension = rank(M)

proof: Since $1\otimes m = (r/r)\otimes m = (1/r)\otimes rm = 0$, if $rm = 0$, we know tensoring with **Q** kills torsion, so $\mathbf{Q}\otimes(\mathbf{Z}^n\oplus T) \cong \mathbf{Q}\otimes\mathbf{Z}^n \cong \mathbf{Q}^n$. QED.

We obtain now an easy proof of an earlier exercise

Corollary: If B is a fin. gen. **Z**-module, and $A\subset B$ a submodule, then $\text{rank}(B/A) = \text{rank}(B) - \text{rank}(A)$.

proof: From the exactness of $0 \to A \to B \to A/B \to 0$, we get the exact sequence $0 \to \mathbb{Q}\otimes A \to \mathbb{Q}\otimes B \to \mathbb{Q}\otimes(A/B) \to 0$, which by the lemma, gives the exact sequence $0 \to \mathbb{Q}^{rk(A)} \to \mathbb{Q}^{rk(B)} \to \mathbb{Q}^{rk(A/B)} \to 0$. Since this sequence of vector spaces splits, $rk(B) = rk(A) + rk(A/B)$. **QED.**

Exercise #184) If $0 \to A \to B \to C \to 0$ is "split exact" then for every X, prove the sequence $0 \to X\otimes A \to X\otimes B \to X\otimes C \to 0$ is exact.

Remarks: (i) If R is a domain, with quotient field k, then k is always a flat R-module.
(ii) If R is any ring, an R-module X is flat iff for every ideal $\alpha \subset R$, the map $\alpha\otimes X \to X$, such that $a\otimes x \mapsto ax$, is injective, i.e. iff tensoring the sequence $0 \to \alpha \to R$ with X, leaves it exact.
(iii) For a nice elementary treatment of flatness, see Lang's <u>Algebra</u>.

Exercise #185) (i) Assuming the previous remark (i), prove that if F,G are free R-modules, R a domain, and $F \subset G$ then $rk(F) \le rk(G)$.
(ii) Assuming the previous remark (ii), prove that if R is a p.i.d., then X is flat/R iff X is torsion-free.

Change of Rings ("Base Change")
We know that if R, S are rings, M is an S-module, and $R \to S$ is a ring map, then M is an R-module by composing $R \to S \to End_{\mathbb{Z}}(M)$. Equivalently, to multiply elements of M by an element r of R, just map r into S then multiply. Tensor products allow us to go in the other direction, and change R modules also into S modules as follows: If M is an R-module, consider the R-module $M\otimes_R S$, which makes sense because S is an R-module via the ring map $R \to S$. Then $M\otimes_R S$ is naturally also an S module where we multiply on the right by S. I.e. for each element σ of S, we define the function $M\times S \to M\otimes S$ where $(m,s) \mapsto m\otimes s\sigma$. This is bilinear, hence defines a linear map $\tilde{\sigma}: M\otimes_R S \to M\otimes_R S$. The map $S \to End(M\otimes_R S)$ sending σ to $\tilde{\sigma}$, takes addition to addition, multiplication to composition, and 1 to id, hence defines a ring map, and even an R-algebra map, hence gives an S-module structure on $M\otimes_R S$ compatible with the R-module structure. What do we know about this S-module? As we have emphasized many times, the most important thing to know is how to map it into other S-modules, so we prove:

Lemma: Given a ring map $R \to S$, an R-module M, and an S-module

N, the map $\mathrm{Hom}_R(M,N) \to \mathrm{Hom}_S(M\otimes_R S,N)$ taking f to $\tilde{f}$ where $\tilde{f}(m\otimes s)=sf(m)$, is an isomorphism (of S-modules).
proof: The inverse map takes g to $\bar{g}$ where $\bar{g}(m) = g(m\otimes 1)$. Note both modules of homomorphisms are S-modules where we multiply by elements of S in the target module. **QED.**

Example: If M = R, both modules in the lemma are isomorphic to N, i.e. $N \cong \mathrm{Hom}_R(R,N) \cong \mathrm{Hom}_S(R\otimes_R S,N) \cong \mathrm{Hom}_S(S,N) \cong N$.

Example: The most important special case may be where M is a free R-module, so we remark then $M\otimes_R S$ is also a free S-module. I.e. if $M \cong \oplus_A R$, then $M\otimes_R S \cong \oplus_A(R\otimes_R S) \cong \oplus_A S$. In particular, if $\{x_i\}_A$ is an R-basis of M, then $\{x_i\otimes 1\}_A$ is an S-basis of $M\otimes_R S$. In particular the isomorphism $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}^m \cong \mathbb{Q}^m$, is one of $\mathbb{Q}$-vector spaces as well as $\mathbb{Z}$-modules. Similarly $\mathbb{C}\otimes_{\mathbb{R}}\mathbb{R}^m \cong \mathbb{C}^m$ is an isomorphism of $\mathbb{C}$-vector spaces.

Remark: Change of rings is "transitive", i.e. if $R\to S\to \tilde{S}$ are ring maps, and M is an R-module, then $M\otimes_R\tilde{S} \cong (M\otimes_R S)\otimes_S\tilde{S}$.

Terminology: Since the ring acting on a module is also known as the "base ring", the process described in this section is sometimes called "base change".

Categorical Sums of Commutative Rings and Algebras
As an extension of the ideas of the section above on base change, consider what happens if *both* modules in a tensor product are rings, hence R-algebras, rather than just R-modules. Let S, T be R-algebras, i.e. let ring maps $\varphi{:}R\to S$, $\psi{:}R\to T$ be given, and form the R-module $S\otimes_R T$ This is both an S-module and a T-module, but we claim it is also a ring, and an R-algebra. The multiplication is the obvious one, i.e. $(s\otimes t)(\tilde{s}\otimes\tilde{t}) = s\tilde{s}\otimes t\tilde{t}$.
Claim: This gives an associative, distributive operation, with identity $1\otimes 1$. First we check it gives a well defined R-bilinear operation: The function $(S\times T)\times(S\times T)\to S\otimes T$, taking $((s,t),(\tilde{s},\tilde{t}))\mapsto s\tilde{s}\otimes t\tilde{t}$ gives, for each fixed value of $(\tilde{s},\tilde{t})$, a bilinear map on $S\times T$, hence induces a linear map $(S\otimes T)\times\{(\tilde{s},\tilde{t})\}\to S\otimes T$. The induced pairing $(S\otimes T)\times(S\times T)\to S\otimes T$ is also bilinear in the second variable for each fixed element of $S\otimes T$, hence induces a map $(S\otimes T)\times(S\otimes T)\to S\otimes T$,

which is linear in each variable. Hence our propsed multiplication is well defined and R-bilinear.
Since $(1\otimes 1)(s\otimes t) = s\otimes t$, the element $1\otimes 1$ acts as an identity on a set of generators, hence also everywhere. Similarly, $(s_0\otimes t_0)(s_1s_2\otimes t_1t_2) = (s_0s_1s_2\otimes t_0t_1t_2) = (s_0s_1\otimes t_0t_1)(s_2\otimes t_2)$, so the product is associative on generators. Since these expressions are linear in each quantity $s_i\otimes t_i$, associativity holds for all elements.
Since the R-module structures on S,T are by means of the maps $\varphi:R\to S$, and $\psi:R\to T$, the following elements of $S\otimes T = S\otimes_R T$ are equal: $r(x\otimes y) = (rx\otimes y) = (\varphi(r)x)\otimes y = x\otimes(\psi(r)y) = (x\otimes ry)$. Thus there is a unique R-algebra structure on $S\otimes T$ defined by the map $R\to S\otimes T$, taking r to $r(1\otimes 1) = 1r\otimes 1 = \varphi(r)\otimes 1 = 1\otimes\psi(r) = 1\otimes r1$. Since $r\tilde{r}(1\otimes 1) = \varphi(r)\otimes\psi(\tilde{r}) = (\varphi(r)\otimes 1)(1\otimes\psi(\tilde{r})) = (r(1\otimes 1))(\tilde{r}(1\otimes 1))$, and $(r+\tilde{r})(1\otimes 1) = r(1\otimes 1) + \tilde{r}(1\otimes 1)$, and $1\mapsto(1\otimes 1)$, this is indeed a ring map.
Remark: With the understanding given above of the notation, we may write simply $r\otimes 1$ for $r(1\otimes 1) = \varphi(r)\otimes 1 = 1\otimes\psi(r)$.

This simple construction yields a nice conclusion:
Theorem: Any two R-algebras $R\to S$, $R\to T$, have a direct sum Γ in the category of R-algebras. In fact, $\Gamma \cong S\otimes_R T$.

Remark: The theorem says that in the category of R-algebras, Γ represents the functor $\mathrm{Hom}_R(S,\cdot)\times\mathrm{Hom}_R(T,\cdot):\mathcal{G}_R\to\mathfrak{M}_R$, from R-algebras to R-modules. I.e. if $\Gamma = S\otimes_R T$, there are R-algebra maps $\sigma:S\to\Gamma$, $\tau:T\to\Gamma$, such that for every R-algebra Λ, the correspondence taking $f:\Gamma\to\Lambda$ to the pair $(\sigma^* f,\tau^* f)$, is a bijection $\mathrm{Hom}_R(\Gamma,\Lambda)\to\mathrm{Hom}_R(S,\Lambda)\times\mathrm{Hom}_R(T,\Lambda)$ from R-algebra maps out of Γ to pairs of R-algebra maps out of S and T.

proof of theorem: Let $\Gamma = S\otimes_R T$. The R-algebra maps $S\to S\otimes T$, $T\to S\otimes T$ are the obvious ones, $s\mapsto s\otimes 1$, and $t\mapsto 1\otimes t$. Since the class of R-algebra maps is closed under composition, any map $f:S\otimes T\to\Lambda$ yields by composition maps $S\to\Lambda$, $T\to\Lambda$, and since $S\otimes T$ is generated by elements of form $s\otimes t$, these compositions determine f.
Conversely, given two maps $g:S\to\Lambda$, $h:T\to\Lambda$, their product gives a bilinear function $S\times T\to\Lambda$, hence a unique R-module map $g\otimes h$: $S\otimes T\to\Lambda$ taking $s\otimes t\mapsto g(s)h(t)$. Since this map takes $(s\otimes t)(\tilde{s}\otimes\tilde{t}) = (s\tilde{s}\otimes t\tilde{t})\mapsto g(s\tilde{s})h(t\tilde{t}) = g(s)h(t)g(\tilde{s})h(\tilde{t})$, it is a ring map. Since it takes

$r(s\otimes t) = rs\otimes t$ to $g(rs)h(s) = rg(s)h(s)$, and g,h are R-algebra maps, $g\otimes h$ is an R-algebra map too. Finally, restricting $g\otimes h$ to elements of form $s\otimes 1$ or $1\otimes t$ gives the original maps g,h. Thus the correspondence is indeed a bijection. QED.

Example: If S,T are commutative rings, the tensor product $S\otimes_{\mathbb{Z}}T$, is a direct sum in the category of commutative rings.

The theorem lets us calculate a particularly useful example of a tensor product of R algebras.
Example: If $S = R[X_1,\ldots,X_n]$, and $T = R[Y_1,\ldots,Y_m]$, are polynomial algebras, then $R[X_1,\ldots,X_n]\otimes_R R[Y_1,\ldots,Y_m] \cong R[X_1,\ldots,X_n,Y_1,\ldots,Y_m]$. To see this we will check that they represent equivalent functors. For brevity, denote the set of all $\{X_i\}$ simply by X and the set of the $\{Y_j\}$ simply by Y. Since R-algebra maps out of a polynomial ring are equivalent to set functions out of the set of variables, we have equivalences: $Hom_R(R[X,Y],\cdot) \cong Hom_{\mathcal{S}}(\{X,Y\},\cdot) \cong Hom_{\mathcal{S}}(X,\cdot)\times Hom_{\mathcal{S}}(Y,\cdot) \cong Hom_R(R[X],\cdot)\times Hom_R(R[Y],\cdot) \cong Hom_R(R[X]\otimes R[Y],\cdot)$. Since $R[X,Y]$ and $R[X]\otimes_R R[Y]$ represent equivalent functors, they are isomorphic. QED.

Example: In algebraic geometry, the structure of an affine algebraic scheme is contained in its structure ring, and for the standard coordinate scheme $\mathbb{C}^n$ that ring is $\mathbb{C}[X_1,\ldots,X_n]$. Since the functor taking an affine scheme to its structure ring is a contravariant equivalence, hence changes products into sums, the previous example implies the product of $\mathbb{C}^n$ and $\mathbb{C}^m$ exists as affine scheme/$\mathbb{C}$, and is isomorphic to $\mathbb{C}^{n+m}$. The theorem itself implies that every pair X, Y of affine R-schemes, with structure rings $\Gamma(X)$, $\Gamma(Y)$, has a direct product with structure ring $\Gamma(X)\otimes_R\Gamma(Y)$.

Tensor Products of Vector Spaces

The most important special case of tensor products are those of finite dimensional vector spaces. We want to focus on that case now. Everything in this section is valid for finite free modules over a (commutative) ring. Since tensor products are a tool for discussing multilinear maps, we begin again with that concept. Throughout this section the base field is fixed, and denoted by k.

Recall: For a finite dimensional vector space M over a field k, let $\mathcal{L}^s(M;R) = \mathcal{L}^s(M) = \{$set of s-multilinear maps $f: M\times\ldots\times M = M^s \to k\}$, where $M\times \ldots \times M = M^s$ denotes the s-fold Cartesian product of M with itself. If N is another finite dimensional k-vector space, $\mathcal{L}^s(M,N)$ is the set of s-linear maps $f: M\times\ldots\times M \to N$.

Remarks (i) Both sets $\mathcal{L}^s(M)$, $\mathcal{L}^s(M;N)$ are naturally k-vector spaces with the usual pointwise operations on functions.
(ii) As in the previous sections, there are natural isomorphisms $\mathcal{L}^s(M) \cong (M\otimes\ldots\otimes M)^* = (M^{\otimes s})^*$ where $(M\otimes\ldots\otimes M) = M^{\otimes s}$ is the s-fold tensor product of M, and $\mathcal{L}^s(M;N) \cong \mathrm{Hom}(M^{\otimes s},N) \cong (M^{\otimes s})^*\otimes N \cong \mathcal{L}^s(M)\otimes N$. For example, if f is in $\mathcal{L}^s(M)$ and n is in N, then $f\otimes n$ in $\mathcal{L}^s(M)\otimes N$ corresponds to the element of $\mathcal{L}^s(M;N)$ taking $\underline{m} = (m_1, \ldots, m_s)$ to $f(\underline{m})\cdot n$. [Write down the other maps as exercise.] Eg. for $s = 1$, $\mathcal{L}^1(M) = M^* =$ the "dual space" of M, and $\mathcal{L}^1(M;N) = \mathrm{Hom}(M,N)$, exhibiting again the isomorphism $M^*\otimes N \cong \mathrm{Hom}(M,N)$.
(iii) If $0\to X\to Y\to Z\to 0$ is an exact sequence of vector spaces, for any vector space M, the sequence $0\to M\otimes X\to M\otimes Y\to M\otimes Z\to 0$ is also exact, (since every module M is free hence also flat over the field k).
(iv) In a sense, tensor products are superfluous in the category of finite dimensional vector spaces; i.e. the isomorphisms $X\otimes Y \cong \mathrm{Hom}(X^*,Y)$ define an equivalence $X\otimes(\cdot) \cong \mathrm{Hom}(X^*,\cdot)$ revealing the tensor product in this case as equivalent to a covariant Hom functor. This is not as surprising if we recall that the Eilenberg-Watts theorem says every right exact linear functor F that commutes with direct sums is equivalent to $F(k)\otimes(\cdot)$. In the category of finite dimensional vector spaces, $\mathrm{Hom}(X^*,\cdot)$ is right exact as well as left exact, and commutes with finite direct sums since those coincide with finite direct products. Thus $\mathrm{Hom}(X^*,\cdot)$ may be expected to be equivalent to $\mathrm{Hom}(X^*,k)\otimes(\cdot) = X^{**}\otimes(\cdot) \cong X\otimes(\cdot)$.

Exercise #186) If F is any linear functor that commutes with finite direct sums on the category of finite dimensional vector spaces over k, prove $F(\cdot) \cong F(k)\otimes_k(\cdot)$ are equivalent functors. [Hint (if desired): If x is an element of X, denote by $\lambda_x: k\to X$ the unique map taking $r\mapsto rx$. Then F gives a map $F(\lambda_x): F(k)\to F(X)$, and we can define the map $\Theta: F(k)\otimes X\to F(X)$ taking $\alpha\otimes x$ to $F(\lambda_x)(\alpha)$, once we note the expression $F(\lambda_x)(\alpha)$ is bilinear in (α,x). To see Θ is an isomorphism (and that it is linear) show Θ is the composition of the isomorphisms $F(k)\otimes X \to F(k)\otimes k^n \to (F(k))^n \to F(k^n) \to F(X)$, induced by the coordinate isomorphism $\varphi: X \to k^n$ associated to any choice of basis of X. It may help to note that the map $\lambda_a: k \to k^n$, taking $r \mapsto ra = (ra_1,\dots,ra_n)$, equals $\Sigma a_i\sigma_i$ where $\sigma_i k \to k^n$ is the injection onto the ith coordinate, i.e. where $\sigma_i(r) = re_i$; that the isomorphism $(F(k))^n \to F(k^n)$ is $\Sigma_i F(\sigma_i)$, and that $\varphi^{-1}\circ\Sigma a_i\sigma_i = \lambda_x$, if $(a_1,\dots,a_n) = \varphi(x)$ is the coordinate vector of x in k^n. In particular, if $F(X) = Hom(M,X)$, then $F(k) = M^*$ and the map $M^*\otimes X \to Hom(M,X)$ is the usual isomorphism taking $\alpha\otimes x$ to the map $\alpha\cdot x: M \to X$ whose value on m is $\alpha(m)\cdot x$.]

Next we lighten up on the abstraction and look again at the very concrete "tensor spaces" $\mathcal{L}^s$ and their relations with one another.

Theorem: For each s, t, there is a well defined multiplication $\otimes: \mathcal{L}^s(M)\times\mathcal{L}^t(M) \to \mathcal{L}^{s+t}(M)$, whose image generates $\mathcal{L}^{s+t}(M)$. In fact, if $\{f_i\}$ is a basis for $\mathcal{L}^s(M)$, and $\{g_j\}$ is a basis for $\mathcal{L}^t(M)$, then the products $\{f_i\otimes g_j\}$ are a basis for $\mathcal{L}^{s+t}(M)$, and this multiplication induces an isomorphism $\mathcal{L}^s(M)\otimes\mathcal{L}^t(M) \to \mathcal{L}^{s+t}(M)$.

proof: If f is in $\mathcal{L}^s(M)$, and g is in $\mathcal{L}^t(M)$, let $(f\otimes g)(v_1,\dots,v_s,w_1,\dots,w_t) = f(v_1,\dots,v_s)g(w_1,\dots,w_t)$. We already know, from the section on tensor products of homomorphisms above, that this is multilinear and induces the natural isomorphisms $\mathcal{L}^s(M)\otimes\mathcal{L}^t(M) \cong (M^{\otimes s})^*\otimes(M^{\otimes t})^* \cong (M^{\otimes s}\otimes M^{\otimes t})^* \cong (M^{\otimes s+t})^* \cong \mathcal{L}^{s+t}(M)$. QED.

Corollary: The dimension of $\mathcal{L}^s(M)$ as a vector space is $(\dim(M))^s$.

proof: Induction on s; if $\dim \mathcal{L}^1(M) = \dim(M^*) = \dim(M) = n$, then $\dim(\mathcal{L}^2) = \dim(\mathcal{L}^1\otimes\mathcal{L}^1) = n^2,\dots, \dim(\mathcal{L}^{s+1}) = \dim(\mathcal{L}^s\otimes\mathcal{L}^1) = n^s n = n^{s+1}$. [Of course we also know this from the isomorphism $\mathcal{L}^s(M) \cong (M^{\otimes s})^*$.] QED.

Remark: **(i)** In particular, if M has dimension n, and if $\{f_1,\dots,f_n\}$ is a basis of M^*, then the set of n^s functions $\{f_{i(1)}\otimes\dots\otimes f_{i(s)}\}$ where i ranges over all functions from the set $\{1,\dots,s\}$ to the set $\{1,\dots,n\}$, forms a basis for $\mathcal{L}^s(M)$. Consequently it is perfectly correct in this setting to think of an element of $\mathcal{L}^s(M)$ as a linear combination of

expressions of form $f_1 \otimes \ldots \otimes f_s$, where the f_i are linear functions on M.
(ii) The natural "evaluation" pairing $(M^{\otimes s})^* \times (M^{\otimes s}) \to R$ taking $(f, \underline{m})$ to $f(\underline{m})$, induces a similar dual pairing $\mathcal{I}^s(M) \otimes (M^{\otimes s}) \to R$, and shows we can think of the elements $f_1 \otimes \ldots \otimes f_s$ of $\mathcal{I}^s(M)$ as acting on elements of form $m_1 \otimes \ldots \otimes m_s$ of $M^{\otimes s}$. (More generally, an element $\Sigma f_1 \otimes \ldots \otimes f_s$ of $\mathcal{I}^s(M)$, acts on an element $\Sigma m_1 \otimes \ldots \otimes m_s$ of $M^{\otimes s}$.)

Corollary: The theorem above shows there is a natural (non commutative) algebra structure on the (infinite dimensional) direct sum vector space $\oplus_{s \geq 0} \mathcal{I}^s(M)$. Here $\mathcal{I}^0(M) = R$. This algebra is isomorphic to the obvious algebra structure on the direct sum $\oplus_{s \geq 0}(M^*)^{\otimes s}$, induced by the bilinear maps $(M^*)^{\otimes s} \otimes (M^*)^{\otimes t} \to (M^*)^{\otimes s+t}$ taking $(f_1 \otimes \ldots \otimes f_s) \otimes (\tilde{f}_1 \otimes \ldots \otimes \tilde{f}_t)$ to $(f_1 \otimes \ldots \otimes f_s \otimes \tilde{f}_1 \otimes \ldots \otimes \tilde{f}_t)$, where $(M^*)^{\otimes 0} = R$, $(M^*)^{\otimes 1} = M^*$.

Remarks: (i) The direct sum vector space $\oplus_{s \geq 0}(M^{\otimes s})$ also has the obvious (non commutative) algebra structure, induced by the bilinear maps $(M^{\otimes s}) \otimes (M^{\otimes t}) \to (M^{\otimes s+t})$ taking $(m_1 \otimes \ldots \otimes m_s) \otimes (\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_t)$ to $(m_1 \otimes \ldots \otimes m_s \otimes \tilde{m}_1 \otimes \ldots \otimes \tilde{m}_t)$. This is the "largest possible" R-algebra generated by M, in the sense that if S is any R-algebra equipped with an R-module map $\alpha: M \to S$ whose image generates S as an R-algebra, then α extends uniquely to a surjective R-algebra map $\oplus_{s \geq 0}(M^{\otimes s}) \to S$.
(ii) $\mathcal{I}^k(M)$ is sometimes called the space of "k-tensors" on M. For historical reasons, and somewhat unfortunately from our point of view, differential geometers call sections of the bundle of these spaces "covariant tensors", although $\mathcal{I}^s$ is a contravariant functor of M. (They are apparently referring to the way the local coordinates of sections of these bundles transform under coordinate change.)

Contraction and Trace

The isomorphism $M^* \otimes M \cong \mathrm{Hom}(M,M)$ offers a new interpretation of a familiar invariant of an endomorphism. I.e. there is an obvious linear map defined on the the object on the left, namely the map $M^* \otimes M \to k$ induced by the evaluation pairing $M^* \times M \to k$ taking $(f,m) \mapsto f(m)$. The induced map $M^* \otimes M \to k$ taking $f \otimes m \mapsto f(m)$, is often called "contraction". That means there is some corresponding natural linear map $\mathrm{Hom}(M,M) \to k$. What is it? Let's compute in coordinates with respect to some basis of M, say $\{x_i\}$. This gives the corresponding dual basis $\{f_j\}$ of M^*, and the basis $\{f_j \otimes x_i\}$ of $M^* \otimes M$, so that a typical element of $M^* \otimes M$ has form $\Sigma a_{ij}\, f_j \otimes x_i$. Recall the isomorphism with Hom(M,M) takes $\Sigma a_{ij}\, f_j \otimes x_i$ to the map T sending m to $T(m) = \Sigma_{ij}\, a_{ij}\, f_j(m) x_i$. The matrix [T] of this element has by definition entry (s,t) = the sth coordinate of $T(x_t)$. Since $T(x_t) = \Sigma_{ij}\, a_{ij}\, f_j(x_t) x_i = \Sigma_i\, a_{it}\, f_t(x_t) x_i = \Sigma_i\, a_{it}\, x_i$, the sth coordinate of this vector (with respect to the basis $\{x_i\}$) is a_{st}. I.e. the matrix [T] = $[a_{ij}]$. Now consider contraction of the element $\Sigma a_{ij}\, f_j \otimes x_i$, which yields $\Sigma_{ij}\, a_{ij}\, f_j(x_i) = \Sigma_i\, a_{ii}$. Thus the composition $\mathrm{Hom}(M,M) \cong M^* \otimes M \to k$, takes T to $\Sigma_i\, a_{ii}$ = "trace" [T]. Thus we get another proof that the trace of the matrix representing an endomorphism is an invariant of the endomorphism itself, independent of the choice of basis and the associated matrix. Equivalently, similar matrices $A \sim B^{-1}AB$, have the same trace.

Exercise #187) Let λ, v be elements of $(k^n)^*$, and k^n respectively, with λ a row vector (1×n matrix), and v a column vector (n×1 matrix). Show that the matrix product $\lambda \cdot v$ is the contraction of the element $\lambda \otimes v$ of $(k^n)^* \otimes k^n$, while the matrix product $v \cdot \lambda$ is the matrix of the homomorphism corresponding to $\lambda \otimes v$ in $\mathrm{Hom}(k^n, k^n)$, i.e. $v \cdot \lambda$ is the "tensor product" of the matrices λ,v as defined above. In particular, $\mathrm{trace}(\lambda \cdot v) = \mathrm{trace}(v \cdot \lambda)$.

Examples: We have encountered some important examples of tensors, i.e. of multilinear maps, in our calculus and linear algebra courses: we have observed that the inner product <·,·> of two vectors in $\mathbb{R}^n$ is a bilinear function, hence an element of $\mathcal{I}^2(\mathbb{R}^n)$; and the "triple product" taking a triple of vectors (u,v,w) in $\mathbb{R}^3$ to the scalar $u \cdot (v \times w)$, where $v \times w$ denotes the "vector cross product", belongs to $\mathcal{I}^3(\mathbb{R}^3)$ Another important tensor, of "mixed type", is the "curvature" tensor R for a surface in space. If T_p denotes the tangent space to the surface at p, R is a section of the "bundle" of spaces $T_p^* \otimes T_p^* \otimes T_p^* \otimes T_p$, hence is called a tensor of type (3,1). This is treated in differential geometry books, [eg. Comprehensive Introduction to Differential geometry, vol.2, by Michael Spivak]. In $\mathcal{I}^n(\mathbb{R}^n)$ there is the determinant, a function of n variables, for instance as a function of the rows of an $n \times n$ matrix, and which is linear in each row separately. The determinant has another property not shared by most other tensors, that of being alternating. We want next to study tensors with this property more closely.

Alternating tensors

Definition: An s-linear function f in $\mathcal{L}^s(M)$ or $\mathcal{L}^n(M,N)$ is called "alternating" iff $f(x_1,\dots,x_s) = 0$ whenever $x_i = x_j$ for some $i \neq j$.

Exercise #188) (i) Prove that if f is alternating, then interchanging any two variables changes the sign of f For example if $s = 2$, and f is alternating, then $f(x,y) = - f(y,x)$.
(ii) Prove conversely that if f changes sign whenever two variables are interchanged, then f is alternating, unless charac(k) = 2.
(iii) What about statement (ii) in characteristic 2?

Notation: The subspaces of alternating s-linear functions are denoted $\Omega^s(M) \subset \mathcal{L}^s(M)$ and $\Omega^s(M;N) \subset \mathcal{L}^s(M,N)$.

Remark: The determinant is an element of $\Omega^n(k^n)$.

Exercise #189): **(i)** Prove that a finite product of linear maps followed by an alternating multilinear map is alternating and multilinear, i.e if $T_\alpha: V_\alpha \to W_\alpha$ are linear for all α, then $(\lambda \circ \prod T_\alpha): \prod V_\alpha \to \prod W_\alpha \to Z$ is alternating multilinear, provided $\lambda. \prod W_\alpha \to Z$ is so.
(ii) From part (i), if $\pi_\alpha : k^n \to k$ is projection on the αth factor, and for $1 \leq \alpha(1) < \alpha(2) < \dots < \alpha(s) \leq n$, if $\alpha = (\alpha(1), \alpha(2),\dots,\alpha(s))$ and $\pi_\alpha = \prod_{i=1\dots s} \pi_{\alpha(i)} : k^n \to k^s$ is the projection $\pi_\alpha : (x_1,\dots,x_n) \mapsto (x_{\alpha(1)},\dots,x_{\alpha(s)})$, then the composition $f_\alpha = \det \circ (\pi_\alpha)^s : (k^n)^s \to (k^s)^s \to k$, is an s-multilinear map $f_\alpha : (k^n) \to k$, where the notation π^s means $\prod_{i=1\dots s} \pi$. Show the set of functions $\{f_\alpha\}$, for all possible values of α are independent, by letting f_α act on $e_\beta = (e_{\beta(1)}, \dots, e_{\beta(s)})$, where $1 \leq \beta(1) < \beta(2) < \dots < \beta(s) \leq n$, and observe that $f_\alpha(e_\beta) = 0$ unless $\alpha(i) = \beta(i)$ for all $i = 1,\dots,s$.

We compute next "all" alternating tensors, and see in particular that the determinant is the only essential one, underlying all others.

Theorem: (i) If $s > n$, then $\Omega^s(k^n) = 0$.

(ii) If $s \le n$, then $\dim \Omega^s(k^n) = \binom{n}{s}$, (binomial coefficient).

(iii) In particular $\dim \Omega^n(k^n) = 1$, and the determinant function is a basis of $\Omega^n(k^n)$.

proof: (i) If f is any s-multilinear function on k^n, and we expand each vector x_j in k^n using the standard basis $\{e_i\}$, we can write $f(x_1,\dots,x_s) = f(\Sigma a_i e_i, \Sigma b_j e_j, \dots, \Sigma c_t e_t) = \Sigma a_i b_j \dots c_t f(e_i,e_j,\dots,e_t)$, as a linear combination of terms, such that each term has a factor of $f(e_i,e_j,\dots,e_t)$, i.e. of f acting on a sequence of s basis vectors. When $s > n$, at least one basis vector e_α must be repeated, hence $f(e_i,e_j,\dots,e_t) = 0$, and thus $f(x_1,\dots,x_s) = 0$ for every s-tuple of vectors $(x_1,\dots,x_s)$ in $(k^n)^s$.

(ii) Since the functions $f_\alpha = \det\circ(\pi_\alpha)^s:(k^n)^s \to k$ in the previous exercise are independent, and there are $\binom{n}{s}$ possible values for α, we get $\dim \Omega^s(k^n) \ge \binom{n}{s}$. To see the opposite inequality, consider the linear map $\Theta:\Omega^s(k^n) \to k^{\binom{n}{s}}$, $f \mapsto (\dots,f(e_\beta),\dots)$, defined by evaluating at all the various s-tuples $e_\beta = (e_{\beta(1)},\dots,e_{\beta(s)})$, for all order preserving injections $\beta:\{1,\dots,s\} \to \{1,\dots,n\}$. It follows from the argument in (i) that f is completely determined if we know the values $f(e_i,e_j,\dots,e_t)$ of f on every sequence of distinct standard basis vectors. On the other hand every such sequence $(e_i,e_j,\dots,e_t)$ is a rearrangement of a sequence $(e_{\beta(1)},\dots,e_{\beta(s)})$ in which the indices are increasing, $1 \le \beta(1) < \beta(2) < \dots < \beta(s) \le n$. Since f is alternating, knowing $f(e_{\beta(1)},\dots,e_{\beta(s)})$ also determines f on any rearrangement of the basis vectors $(e_{\beta(1)},\dots,e_{\beta(s)})$. Thus $\Theta(f)$ determines all values of f and Θ is injective.. Hence $\dim \Omega^s(k^n) \le \binom{n}{s}$, and thus equality holds.

(iii) This follows from (ii). QED.

Summary: To construct a basis element of $\Omega^s(k^n)$, where $s \le n$, just choose one of the $s \times s$ subdeterminants of the $n \times s$ matrix formed by s ordered vectors of k^n.

Example: Suppose $f:(k^3)^2 \to k$ is an element of $\Omega^2(k^3)$. Then for $x = a_1e_1+a_2e_2+a_3e_3$, and $y = b_1e_1+b_2e_2+b_3e_3$, we have $f(x,y) = \Sigma_{i,j}\, a_ib_jf(e_i,e_j) = \Sigma_{i\neq j}\, a_ib_jf(e_i,e_j) = \Sigma_{i<j}\, (a_ib_jf(e_i,e_j) + a_jb_if(e_j,e_i)) = \Sigma_{i<j}\, (a_ib_jf(e_i,e_j) - a_jb_if(e_i,e_j)) = \Sigma_{i<j}\, (a_ib_j - a_jb_i)f(e_i,e_j) = (a_1b_2 - a_2b_1)f(e_1,e_2) + (a_1b_3 - a_3b_1)f(e_1,e_3) + (a_2b_3 - a_3b_2)f(e_2,e_3)$. Thus if the coefficient vectors of x and y are arranged as the column vectors of a 3×2 matrix, there are three obvious alternating functions, the three 2×2 subdeterminants $(a_1b_2 - a_2b_1)$, $(a_1b_3 - a_3b_1)$, and $(a_2b_3 - a_3b_2)$. Any alternating f is a linear combination of those, with coefficients $f(e_1,e_2)$, $f(e_1,e_3)$ and $f(e_2,e_3)$.

Remarks: Just as the space of s-multilinear functions $\mathcal{L}^s(M)$ is dual to the space $M^{\otimes s}$, the subspace $\Omega^s(M) \subset \mathcal{L}^s(M)$ of alternating s-linear functions is dual to a quotient space $\Lambda^s(M)$ of $M^{\otimes s}$. Formally, the elements of $\Lambda^s(M)$ are linear combinations of the symbols $x_1\wedge x_2\wedge x_3 \ldots \wedge x_s$ for all $x_1,x_2,x_3, \ldots ,x_s$ in M, subject to the s-linear alternating "relations". In particular, in $\Lambda^s(M)$, $x_1\wedge x_1\wedge x_3 \ldots\wedge x_s = 0$, $x_1\wedge x_2\wedge x_3 \ldots\wedge x_s = - x_2\wedge x_1\wedge x_3 \ldots\wedge x_s$, etc....

Eg. If $s = 2$, $M = k^3$, $x = a_1e_1+a_2e_2+a_3e_3$, and $y = b_1e_1+b_2e_2+b_3e_3$, then $x\wedge y = (a_1e_1+a_2e_2+a_3e_3)\wedge(b_1e_1+b_2e_2+b_3e_3)$

$= (a_1e_1)\wedge(b_2e_2) + (a_1e_1)\wedge(b_3e_3) + (a_2e_2)\wedge(b_1e_1)$
$\quad + (a_2e_2)\wedge(b_3e_3) + (a_3e_3)\wedge(b_1e_1) + (a_3e_3)\wedge(b_2e_2)$
$= (a_1e_1)\wedge(b_2e_2) + (a_1e_1)\wedge(b_3e_3) - (b_1e_1)\wedge(a_2e_2)$
$\quad + (a_2e_2)\wedge(b_3e_3) - (b_1e_1)\wedge(a_3e_3) - (b_2e_2)\wedge(a_3e_3)$
$= (a_1b_2)(e_1\wedge e_2) + (a_1b_3)(e_1\wedge e_3) - (b_1a_2)(e_1\wedge e_2)$
$\quad + (a_2b_3)(e_2\wedge e_3) - (b_1a_3)(e_1\wedge e_3) - (b_2a_3)(e_2\wedge e_3)$
$= (a_1b_2 - b_1a_2)(e_1\wedge e_2) + (a_1b_3 - b_1a_3)(e_1\wedge e_3) + (a_2b_3 - b_2a_3)(e_2\wedge e_3)$.

(Note that the coefficient functions occurring here are the standard basis elements of $\Omega^2(k^3)$. This reflects an isomorphism $\Omega^2(k^3) \cong (\Lambda^2(k^3))^*$ in which the standard basis of $\Omega^2(M)$ is viewed as dual to the basis $\{e_1\wedge e_2,\ e_1\wedge e_3,\ e_2\wedge e_3\}$ of $\Lambda^2(k^3)$.)

By such computations one can show that if $e_1,\ldots,e_n$ is a basis for M, then the $\binom{n}{s}$ elements $\{e_{\alpha(1)}\wedge\ldots\wedge e_{\alpha(s)},\ 1\le \alpha(1) < \alpha(2) <\ldots< \alpha(s) \le n\}$ span $\Lambda^s(M)$. Since $\dim \Lambda^s(M) = \dim. \Omega^s(M) = \binom{n}{s}$, these elements are also independent.

Our last remark is that for each s, there is a fundamental isomorphism $\oplus_{i+j=s} (\Lambda^i(X) \otimes \Lambda^j(Y)) \cong \Lambda^s(X \oplus Y)$ which can be seen by writing down bases for the two spaces. In particular let $e_1, \ldots, e_n, f_1, \ldots, f_m$ be a basis for $X \oplus Y$, where $e_1, \ldots, e_n$ is a basis for $X \oplus \{0\} \cong X$, and $f_1, \ldots, f_m$ is a basis for $\{0\} \oplus Y \cong Y$. Then a basis for $\Lambda^s(X \oplus Y)$ is given by all elements of form $e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(i)} \wedge f_{\tau(1)} \wedge \ldots \wedge f_{\tau(j)}$ where $i+j = s$, $(i, j \geq 0)$, σ ranges over all strictly increasing functions $\sigma:\{1, \ldots, i\} \to \{1, \ldots, n\}$, and τ ranges over all strictly increasing functions $\tau:\{1, \ldots, j\} \to \{1, \ldots, m\}$. On the other hand, the elements of form $e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(i)} \otimes f_{\tau(1)} \wedge \ldots \wedge f_{\tau(j)}$ where again $i+j = s$, $(i, j \geq 0)$, and σ and τ are as before, give a basis of $\oplus_{i+j=s} (\Lambda^i(X) \otimes \Lambda^j(Y))$. Hence the unique linear map $\oplus_{i+j=s} (\Lambda^i(X) \otimes \Lambda^j(Y)) \to \Lambda^s(X \oplus Y)$ taking $e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(i)} \otimes f_{\tau(1)} \wedge \ldots \wedge f_{\tau(j)}$ to $e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(i)} \wedge f_{\tau(1)} \wedge \ldots \wedge f_{\tau(j)}$ is an isomorphism.

Exercise #190) (i) Compute the constant δ such that $(ae_1 + be_2 + ce_3) \wedge (de_1 + fe_2 + ge_3) \wedge (he_1 + je_2 + ke_3) = \delta(e_1 \wedge e_2 \wedge e_3)$, in $\Lambda^3(k^3)$ by expanding the left hand side according to the rules for manipulating "wedge products" $x \wedge y \wedge z$.
(ii) If $\alpha:\Omega^2(k^3) \to (\Lambda^2(k^3))^*$ is the isomorphism taking the ordered basis $\{(a_1b_2 - b_1a_2), (a_1b_3 - b_1a_3), (a_2b_3 - b_2a_3)\}$ to the basis dual to $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$, show for every f in $\Omega^2(k^3)$ and x, y in k^3, that $\alpha(f)(x \wedge y) = f(x,y)$.
(iii) Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the basis of $(k^3)^*$ dual to $\{e_1, e_2, e_3\}$, and $\beta:\Lambda^2((k^3)^*) \to \Omega^2(k^3)$ the isomorphism taking the ordered basis $\{\lambda_1 \wedge \lambda_2, \lambda_1 \wedge \lambda_3, \lambda_2 \wedge \lambda_3\}$ to the ordered basis $\{(a_1b_2 - b_1a_2), (a_1b_3 - b_1a_3), (a_2b_3 - b_2a_3)\}$. Show for any elements f, g of $(k^3)^*$, and x, y of k^3, that $(\beta(f \wedge g))(x,y) = f(x)g(y) - f(y)g(x)$.
(iv) If $\gamma:\Lambda^3((k^3)^*) \to \Omega^3(k^3)$ is the isomorphism taking $\lambda_1 \wedge \lambda_2 \wedge \lambda_3$ to the 3x3 determinant function, f, g, h are elements of $(k^3)^*$ and x, y, z are elements of k^3, what is $(\gamma(f \wedge g \wedge h))(x \wedge y \wedge z)$?

Example: A smooth "one form" on $\mathbb{R}^2$ is a smooth, i.e. infinitely differentiable, map $\omega:\mathbb{R}^2 \to \Lambda^1(\mathbb{R}^2)^* = (\mathbb{R}^2)^*$. Let dx, dy denote the "constant" one forms, where $dx(p) = \lambda_1$, $dy(p) = \lambda_2$, for all p, with $\{\lambda_1, \lambda_2\}$ the basis of $(\mathbb{R}^2)^*$ dual to $\{e_1, e_2\}$. Then every smooth one form can be written as $\omega = fdx + gdy$, where f,g, are smooth (real

valued) functions on $\mathbb{R}^2$. For each smooth function φ on $\mathbb{R}^2$, define "grad(φ)" = $d\varphi = (\partial\varphi/\partial x)\,dx + (\partial\varphi/\partial y)\,dy$, a smooth one form. For each smooth one form $\omega = f dx + g dy$, define "curl ω" = $d\omega = df \wedge dx + dg \wedge dy$, a smooth two form. If $\gamma(t) = (x(t), y(t))$, $0 \le t \le 1$, is a parametrized arc in $\mathbb{R}^2$, define the integral of ω over γ to be
$\int_\gamma \omega = \int_{t=0}^{t=1} f(\gamma(t))(dx/dt) + g(\gamma(t))(dy/dt)\, dt$.

Exercise #191) (i) Prove for every smooth function φ, that
$\int_\gamma d\varphi = \varphi(\gamma(1)) - \varphi(\gamma(0))$.
(ii) If $\Theta = \arctan(y/x)$, compute $d\Theta$, and curl($d\Theta$).
(iii) If $\gamma(t) = (\cos(2\pi t), \sin(2\pi t))$, $0 \le t \le 1$, compute $\int_\gamma d\Theta$. Does this contradict part (i)? Why or why not?
(iv) Prove for every smooth function φ, that curl(grad(φ)) = 0.

Exercise #192) (i) A linear map $T:M \to N$ of vector spaces induces a linear map $T^n:M^n \to N^n$ with $T^n(x_1,\dots,x_n) = (Tx_1,\dots,Tx_n)$. Define the map $\Omega^s(T):\Omega^s(N) \to \Omega^s(M)$ by setting $\Omega^s(T)(f) = (T^n)^*(f) = f \circ T^n$. Show that this makes Ω^s a contravariant functor of vector spaces.
(ii) If V is a one dimensional k vector space, show the map $k \to \mathrm{Hom}_k(V,V)$ taking λ to $\lambda\cdot(\ )$ = "multiplication by λ", is an isomorphism.
(iii) If $\dim(M) = n$, and $T:M \to M$ is linear, it follows from parts (i) and (ii) that the map $\Omega^n(T)$ corresponds to an element of k under the isomorphism $k \to \mathrm{Hom}_k(\Omega^n(M),\Omega^n(M))$. I.e. the map $\Omega^n(T)$ is simply multiplication by a scalar. Identify this scalar.

This ends our discussion of tensor products specifically of vector spaces. We will give a self contained discussion of $\Lambda^s(M)$ for general modules M in the next section.

§15) Exterior Products of modules

The abstract approach to determinants is to consider the functor of alternating multilinear functions, and then represent that functor, analogous to the way we introduced the tensor product to represent the functor of all multilinear functions. Thus given an R-module X, we want to construct an "exterior product" $M = X \wedge \dots \wedge X = \Lambda^n(X)$, a module such that linear maps out of M are the same as n-linear alternating maps out of X^n (The discussion here will be independent of the brief introductory sketch given above of $\Lambda^n(X)$ in the case of a finite dimensional vector space X. The previous section may be useful as an introduction to the ideas which receive a more abstract treatment here.)

More precisely, given a module X, if $\Omega^n(X;Y)$ denotes n-linear alternating maps $f:X^n \to Y$, we seek a module M such that the functors, $\Omega^n(X;\cdot)$ and $\mathrm{Hom}(M,\cdot)$ are equivalent. To do this we can take advantage of the prior construction of the tensor product. I.e. every n-linear map out of X^n induces a unique linear map F out of $X\otimes....\otimes X = \otimes^n(X)$, and an n-linear map $f:X^n \to Y$ is alternating if and only if the induced map $F:X\otimes....\otimes X \to Y$ vanishes on those elementary tensors $x_1\otimes...\otimes x_n$ in which at least two of the x_i are equal. Hence such maps correspond to linear maps out of the quotient $(X\otimes....\otimes X)/J$ where J is the submodule of $(X\otimes....\otimes X)$ generated by the set of elementary tensors having at least two equal entries. This quotient is our M.

Definition: If M is an R-module, and J is the R-submodule of $(X\otimes....\otimes X)$ defined above, let $(X\otimes....\otimes X)/J = \Lambda^n(X) = X\wedge....\wedge X$, the nth exterior product, or "wedge product", of X, with its natural R-module structure. We note that $\Lambda^1(X) = X$ (since $\Omega^1 = \mathrm{Hom}$), and we agree that $\Lambda^0(X) = R$.

Exercise #193) (i) If we denote by $x_1\wedge....\wedge x_n$, the equivalence class of $x_1\otimes....\otimes x_n$ in $\Lambda^n(X)$, then the canonical map $\Theta:X^n \to \Lambda^n(X)$ taking $(x_1,...,x_n)$ to $x_1\wedge....\wedge x_n$ is n-linear and alternating.

(ii) If $T:\Lambda^n(X) \to Y$ is a linear map, the composition $\Theta^*(T) = (T\circ\Theta):X^n \to \Lambda^n(X) \to Y$ is n-linear and alternating.

(iii) For a given X, the maps Θ^* define an equivalence of covariant functors $\mathrm{Hom}(\Lambda^n(X),\cdot) \cong \Omega^n(X;\cdot)$.

(iv) If X is a finitely generated R-module with fewer than n generators, prove $\Lambda^n(X) = 0$. [Hint: show $\Omega^n(X;Y)=0$ for all Y.]

Exercise #194) For each n, prove that $X \mapsto \Lambda^n(X)$ defines a functor from R-modules to R-modules. I.e. if $T:X \to Y$ is linear, prove there is a unique well defined linear map $\Lambda^n(T):\Lambda^n(X) \to \Lambda^n(Y)$ such that

$\wedge^n(T)(x_1\wedge\ldots\wedge x_n) = Tx_1\wedge\ldots\wedge Tx_n$, and that this makes $\wedge^n$ into a functor. [To begin, show the submodule J of $(X\otimes\ldots\otimes X)$ in the definition of $\wedge^n(X)$, is in the kernel of the composition $\otimes^n(X)\to\otimes^n(Y)\to\wedge^n(Y)$ taking $x_1\otimes\ldots\otimes x_n$ to $Tx_1\wedge\ldots\wedge Tx_n$.]

The primary computational result about exterior products is the following.

Theorem: For every $s \geq 0$, and all R-modules X and Y, there is an isomorphism $\oplus_{i+j=s}(\wedge^i(X)\otimes\wedge^j(Y)) \cong \wedge^s(X\oplus Y)$ induced by natural linear maps $(\wedge^i(X)\otimes\wedge^j(Y))\to\wedge^{i+j}(X\oplus Y)$ which take $x_1\wedge\ldots\wedge x_i\otimes y_1\wedge\ldots\wedge y_j$ to $x_1\wedge\ldots\wedge x_i\wedge y_1\wedge\ldots\wedge y_j$.

To illustrate how powerful the theorem is, before proving it we will deduce from it the existence and uniqueness of determinants, a formula for them, their multiplicativity property, a computation of the exterior products of all finite free modules and consequently also of the spaces of n-linear alternating maps on finite free modules. First we claim that $\wedge^n(R^n) \cong R$. This is true when $n = 1$, and then by induction and part (iv) of Ex. #181, $\wedge^n(R^n) \cong \wedge^n(R\oplus R^{n-1}) \cong \wedge^1(R)\otimes\wedge^{n-1}(R^{n-1}) \cong R\otimes R \cong R$. This gives uniqueness of the determinant, since $\wedge^n(R^n) \cong R$ implies that $\Omega^n(R^n,R) \cong \mathrm{Hom}(\wedge^n(R^n),R) \cong \mathrm{Hom}(R,R) \cong R$, so the module of n-linear alternating maps $\mathrm{Mat}_n(R) \cong (R^n)^n\to R$ is free of rank one. Thus there is at most one such map with a given value on the identity matrix. Now we can also define the determinant of a map in Hom(X,X) where $X \cong R^n$, without choosing a basis, as the composition $\wedge^n:\mathrm{Hom}(X, X)\to\mathrm{Hom}(\wedge^n(X), \wedge^n(X)) \cong R$; i.e. since $X \cong R^n$ implies $\wedge^n(X) \cong R$, there is a *canonical* isomorphism $\mathrm{Hom}(\wedge^n(X),\wedge^n(X)) \cong R$. Since $\wedge^n$ is a functor (by the previous exercise), the identity endomorphism of X goes to the identity map of $\wedge^n$, which corresponds to 1 in R. Thus there is a unique n-linear alternating map $\mathrm{Mat}_n(R) \cong (R^n)^n\to R$ with value 1 on the identity, namely the determinant D. We also get multiplicativity of the determinant from this, since the functor $\wedge^n$ takes compositions in Hom(X, X) to compositions in $\mathrm{Hom}(\wedge^n(X), \wedge^n(X))$, and the isomorphism $\mathrm{Hom}(\wedge^n(X), \wedge^n(X)) \cong R$ takes composition in $\mathrm{Hom}(\wedge^n(X), \wedge^n(X))$ to multiplication in R.

To use this definition to compute a formula for the determinant is easy as well. In terms of the standard basis $e_1, \ldots, e_n$ for R^n, if $T = (a_{ij})$, then $\Lambda^n(T)(e_1 \wedge \ldots \wedge e_n) = Te_1 \wedge \ldots \wedge Te_n =$ $(a_{11}e_1 + \ldots + a_{n1}e_n) \wedge \ldots \wedge (a_{1n}e_1 + \ldots + a_{nn}e_n) =$ $\Sigma_\sigma (a_{\sigma(1)1}e_{\sigma(1)} \wedge \ldots \wedge (a_{\sigma(n)n}e_{\sigma(n)})$ [summed over all permutations σ of $\{1, \ldots, n\}$] $= \Sigma_\sigma (a_{\sigma(1)1} \ldots a_{\sigma(n)n})(e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(n)}) =$ $\Sigma_\sigma \mathrm{sign}(\sigma)(a_{\sigma(1)1} \ldots a_{\sigma(n)n})(e_1 \wedge \ldots \wedge e_n)$. In particular this computation shows that $e_1 \wedge \ldots \wedge e_n$ generates $\Lambda^n(R^n)$. Hence $D(T) =$ $\Sigma_\sigma\ \mathrm{sign}(\sigma)(a_{\sigma(1)1} \ldots a_{\sigma(n)n})$, a standard formula for the determinant [see the appendix].

The theorem implies further that if $n \geq s$, then $\Lambda^s(R^n) \cong R^{\binom{n}{s}}$, where $\binom{n}{s}$ is the binomial coefficient "n choose s". To prove this by induction on n, we may assume $n > s$. Then $\Lambda^s(R^n) \cong \Lambda^s(R \oplus R^{n-1})$ $\cong [\Lambda^0(R) \otimes \Lambda^s(R^{n-1})] \oplus [\Lambda^1(R) \otimes \Lambda^{s-1}(R^{n-1})] \cong$ $\Lambda^s(R^{n-1}) \oplus [\Lambda^1(R) \otimes \Lambda^{s-1}(R^{n-1})] \cong \Lambda^s(R^{n-1}) \oplus \Lambda^{s-1}(R^{n-1}) \cong$ $R^{\binom{n-1}{s}} \oplus R^{\binom{n-1}{s-1}} \cong R^{\binom{n}{s}}$ by the well known recursive formula for binomial coefficients. It follows then too that $\Omega^s(R^n;R) \cong$ $\mathrm{Hom}(\Lambda^s(R^n),R) \cong R^{\binom{n}{s}}$. More generally, for all Y, $\Omega^s(R^n;Y) \cong$ $\mathrm{Hom}(\Lambda^s(R^n),Y) \cong Y^{\binom{n}{s}}$. If we look at the map defining the isomorphism $\Lambda^s(R^{n-1}) \oplus [\Lambda^1(R) \otimes \Lambda^{s-1}(R^{n-1})] \to \Lambda^s(R^n)$, we can specify a basis of $\Lambda^s(R^n)$. If $s = 0$, $\Lambda^0(M) = R$ and 1 is a basis, so assume $s \geq 1$. If $e_1, \ldots, e_n$ is the usual basis of R^n, then we claim the set $\{e_{\alpha(1)} \wedge \ldots \wedge e_{\alpha(s)}, 1 \leq \alpha(1) < \alpha(2) < \ldots < \alpha(s) \leq n\}$ is a basis of $\Lambda^s(R^n)$. This is true for $n = 1$, so if we assume it for $n-1$, then $\{e_{\alpha(1)} \wedge \ldots \wedge e_{\alpha(s)}, 1 \leq \alpha(1) < \alpha(2) < \ldots < \alpha(s) \leq n-1\}$, and $\{e_{\alpha(1)} \wedge \ldots \wedge e_{\alpha(s-1)}, 1 \leq \alpha(1) < \alpha(2) < \ldots < \alpha(s-1) \leq n-1\}$ are bases of $\Lambda^s(R^{n-1})$ and $\Lambda^{s-1}(R^{n-1})$ respectively. Then the set $\{e_{\alpha(1)} \wedge \ldots \wedge e_{\alpha(s-1)} \otimes e_n, 1 \leq \alpha(1) < \alpha(2) < \ldots < \alpha(s-1) \leq n-1\}$ is a basis of $\Lambda^{s-1}(R^{n-1}) \otimes \Lambda^1(R)$, and therefore the isomorphism $\Lambda^s(R^{n-1}) \oplus [\Lambda^{s-1}(R^{n-1}) \otimes \Lambda^1(R)] \to \Lambda^s(R^n)$ described above takes the basis $\{e_{\alpha(1)} \wedge \ldots \wedge e_{\alpha(s)}, 1 \leq \alpha(1) < \alpha(2) < \ldots < \alpha(s) \leq n-1\}$, $\cup$ $\{e_{\alpha(1)} \wedge \ldots \wedge e_{\alpha(s-1)} \otimes e_n, 1 \leq \alpha(1) < \alpha(2) < \ldots < \alpha(s-1) \leq n-1\}$ of the left hand side, to the set $\{e_{\alpha(1)} \wedge \ldots \wedge e_{\alpha(s)}, 1 \leq \alpha(1) < \alpha(2) < \ldots < \alpha(s) \leq n\}$.

which is thus a basis of $\Lambda^s(R^n)$ as claimed.

proof of theorem: As usual, we simply write down the most natural maps we can think of in both directions, and check they are mutually inverse. The only difficulty, due to the complexity of the modules involved, is to verify the maps are well defined.
To define a map $\oplus_{i+j=s} (\Lambda^i(X)\otimes\Lambda^j(Y))\to\Lambda^s(X\oplus Y)$ means of course to define maps $(\Lambda^i(X)\otimes\Lambda^j(Y))\to\Lambda^{i+j}(X\oplus Y)$ for each i,j. We may identify X and Y with the submodules $X\oplus\{0\}$ and $\{0\}\oplus Y$ of $X\oplus Y$. The only natural map seems to be to send an element such as $x_1\wedge\ldots\wedge x_i\otimes y_1\wedge\ldots\wedge y_j$ to $x_1\wedge\ldots\wedge x_i\wedge y_1\wedge\ldots\wedge y_j$, but we must check that there is a well defined homomorphism that does this.

Lemma: For each i,j there is a natural linear map $(\Lambda^i(X)\otimes\Lambda^j(Y))\to\Lambda^{i+j}(X\oplus Y)$ sending $x_1\wedge\ldots\wedge x_i\otimes y_1\wedge\ldots\wedge y_j$ to $x_1\wedge\ldots\wedge x_i\wedge y_1\wedge\ldots\wedge y_j$.
proof: First we consider the function $X^i\times Y^j\to\Lambda^{i+j}(X\oplus Y)$ sending $(x_1,\ldots,x_i;y_1,\ldots,y_j)$ to $x_1\wedge\ldots\wedge x_i\wedge y_1\wedge\ldots\wedge y_j$. If we fix $y_1,\ldots,y_j$, the restricted map $(x_1,\ldots,x_i)\mapsto(x_1\wedge\ldots\wedge x_i\wedge y_1\wedge\ldots\wedge y_j)$ is i-linear and alternating, hence induces a linear map $\Lambda^i(X)\to\Lambda^{i+j}(X\oplus Y)$ sending $x_1\wedge\ldots\wedge x_i$ to $x_1\wedge\ldots\wedge x_i\wedge y_1\wedge\ldots\wedge y_j$. This defines a function $\Lambda^i(X)\times Y^j\to\Lambda^{i+j}(X\oplus Y)$ which is linear in the first variable. If in this function we fix the first variable equal to say $\omega = \Sigma\lambda_\alpha x_{\alpha,1}\wedge\ldots\wedge x_{\alpha,i}$ in $\Lambda^i(X)$, the resulting function $Y^j\to\Lambda^{i+j}(X\oplus Y)$ taking $(y_1,\ldots,y_j)$ to $\omega\wedge y_1\wedge\ldots\wedge y_j = \Sigma\lambda_\alpha x_{\alpha,1}\wedge\ldots\wedge x_{\alpha,i}\wedge y_1\wedge\ldots\wedge y_j$, is j-linear and alternating, hence induces a linear map $\Lambda^j(Y)\to\Lambda^{i+j}(X\oplus Y)$ which takes $y_1\wedge\ldots\wedge y_j$ to $\omega\wedge y_1\wedge\ldots\wedge y_j$. Thus we have a well defined map $\Lambda^i(X)\times\Lambda^j(Y))\to\Lambda^{i+j}(X\oplus Y)$ taking (ω,ν) to $\omega\wedge\nu$, where if $\omega = \Sigma\lambda_\alpha x_{\alpha,1}\wedge\ldots\wedge x_{\alpha,i}$, and $\nu = \Sigma\mu_\beta y_{\beta,1}\wedge\ldots\wedge y_{\beta,j}$, then $\omega\wedge\nu = \Sigma_{\alpha,\beta}\lambda_\alpha\mu_\beta\, x_{\alpha,1}\wedge\ldots\wedge x_{\alpha,i}\wedge y_{\beta,1}\wedge\ldots\wedge y_{\beta,j}$. Finally, this map is bilinear, hence induces a linear map $(\Lambda^i(X)\otimes\Lambda^j(Y))\to\Lambda^{i+j}(X\oplus Y)$ such that $\omega\otimes\nu$ maps to $\omega\wedge\nu$. ["Naturality" is left to the reader.]
QED lemma.

The lemma gives us a linear map $\varphi:\oplus_{i+j=s} (\Lambda^i(X)\otimes\Lambda^j(Y))\to\Lambda^s(X\oplus Y)$. To define an inverse map $\psi:\Lambda^s(X\oplus Y)\to\oplus_{i+j=s} (\Lambda^i(X)\otimes\Lambda^j(Y))$, we look for a function $(X\oplus Y)^s\to\oplus_{i+j=s} (\Lambda^i(X)\otimes\Lambda^j(Y))$ which is s-linear and alternating. This is a little less obvious, so we consider some small values of s, for guidance.

If s = 1, we have $\Lambda^1(X\oplus Y) = X\oplus Y$, and seek a map
$X\oplus Y\to(\Lambda^1(X)\otimes\Lambda^0(Y))\oplus(\Lambda^0(X)\otimes\Lambda^1(Y)) =$
$(X\otimes R)\oplus(R\otimes Y) \cong X\oplus Y$. Hence the obvious choice is
$(x,y)\mapsto x\otimes 1 + 1\otimes y \simeq (x,y)$

If s = 2, we must define a map
$\Lambda^2(X\oplus Y)\to(\Lambda^2(X))\oplus(\Lambda^1(X)\otimes\Lambda^1(Y))\oplus(\Lambda^2(Y))$. In order to invert the previous map, the element $(x_1+y_1)\wedge(x_2+y_2)$
$= (x_1\wedge x_2)+(x_1\wedge y_2)+(y_1\wedge x_2)+(y_1\wedge y_2)$
$= (x_1\wedge x_2)+(x_1\wedge y_2)-(x_2\wedge y_1)+(y_1\wedge y_2)$, must go to
$(x_1\wedge x_2)+(x_1\otimes y_2)-(x_2\otimes y_1)+(y_1\wedge y_2)$. Does this make sense? I.e. suppose we define $(X\oplus Y)^2\to(\Lambda^2(X))\oplus(\Lambda^1(X)\otimes\Lambda^1(Y))\oplus(\Lambda^2(Y))$ by sending $((x_1+y_1),(x_2+y_2))\mapsto(x_1\wedge x_2)+(x_1\otimes y_2)-(x_2\otimes y_1)+(y_1\wedge y_2)$. This is well defined, since the element $((x_1+y_1),(x_2+y_2))$ determines the components x_1,y_1,x_2,y_2. Moreover this map is bilinear, and if $x_1=x_2$ and $y_1=y_2$, the image is zero. Hence there is a linear map $\Lambda^2(X\oplus Y)\to(\Lambda^2(X))\oplus(\Lambda^1(X)\otimes\Lambda^1(Y))\oplus(\Lambda^2(Y))$ taking $(x_1+y_1)\wedge(x_2+y_2)$ to $(x_1\wedge x_2)+(x_1\otimes y_2)-(x_2\otimes y_1)+(y_1\wedge y_2)$, as desired.

If s = 3, $(x_1+y_1)\wedge(x_2+y_2)\wedge(x_3+y_3)$
$= (x_1\wedge x_2\wedge x_3) + (x_1\wedge x_2\wedge y_3) + (x_1\wedge y_2\wedge x_3) + (x_1\wedge y_2\wedge y_3) +$
$(y_1\wedge x_2\wedge x_3) + (y_1\wedge x_2\wedge y_3) + (y_1\wedge y_2\wedge x_3) + (y_1\wedge y_2\wedge y_3)$, and if we group by the number of x's and y's in each term,
$= (x_1\wedge x_2\wedge x_3)$
$+ (x_1\wedge x_2\wedge y_3) + (x_1\wedge y_2\wedge x_3) + (y_1\wedge x_2\wedge x_3)$
$+ (x_1\wedge y_2\wedge y_3) + (y_1\wedge x_2\wedge y_3) + (y_1\wedge y_2\wedge x_3)$
$+ (y_1\wedge y_2\wedge y_3)$, now if we move all y's to the right of all x's,
$= (x_1\wedge x_2\wedge x_3)$
$+ (x_1\wedge x_2\wedge y_3) - (x_1\wedge x_3\wedge y_2) + (x_2\wedge x_3\wedge y_1)$
$+ (x_1\wedge y_2\wedge y_3) - (x_2\wedge y_1\wedge y_3) + (x_3\wedge y_1\wedge y_2)$
$+ (y_1\wedge y_2\wedge y_3)$.
This element then should be mapped to the element
$(x_1\wedge x_2\wedge x_3)$
$+ (x_1\wedge x_2\otimes y_3) - (x_1\wedge x_3\otimes y_2) + (x_2\wedge x_3\otimes y_1)$
$+ (x_1\otimes y_2\wedge y_3) - (x_2\otimes y_1\wedge y_3) + (x_3\otimes y_1\wedge y_2)$

$+ (y_1 \wedge y_2 \wedge y_3)$,
in $\Lambda^3(X) \oplus (\Lambda^2(X) \otimes \Lambda^1(Y)) \oplus (\Lambda^1(X) \otimes \Lambda^2(Y)) \oplus \Lambda^3(Y)$. Note that this is well defined since the last expression is alternating; eg. it vanishes if say $x_1 = x_2$ and $y_1 = y_2$.

Now define a function $(X \oplus Y)^s \to \oplus_{i+j=s} (\Lambda^i(X) \otimes \Lambda^j(Y))$ generalizing this one as follows. Consider $(x_1+y_1, \ldots, x_s+y_s)$ in $(X \oplus Y)^s$ and map it to the element $\Sigma_\alpha \pm (x_{\alpha(1)} \wedge \ldots \wedge x_{\alpha(i)} \otimes y_{\beta(1)} \wedge \ldots \wedge y_{\beta(j)})$. In this sum the sequence α of indices $1 \le \alpha(1) < \ldots < \alpha(i) \le s$, ranges over all 2^s subsets of the set $\{1, \ldots, s\}$, and for each subset α, $1 \le \beta(1) < \ldots < \beta(j) \le s$ is the complementary subset. The sign of each term is the sign of the permutation needed to transform the sequence $(1, 2, \ldots, s)$ into $(\alpha(1), \ldots, \alpha(i), \beta(1), \ldots, \beta(j))$. This is a well defined function on $(X \oplus Y)^s$ which is easily seen to be s-linear. Note for example, the image element is a sum of terms in each of which either x_1 or y_1 occurs, but not both. Hence every term is linear in (x_1+y_1)
To see why our function is alternating, assume $x_n = x_m$ and $y_n = y_m$, where $m = n+1$. Then all terms in the sum $\Sigma_\alpha \pm (x_{\alpha(1)} \wedge \ldots \wedge x_{\alpha(i)} \otimes y_{\beta(1)} \wedge \ldots \wedge y_{\beta(j)})$ vanish in which both n,m occur among the α's, or in which both occur among the β's. So condsider a term in which n is one of the α's and m is one of the β's. There is a dual term having the same α's and β's, excepting only that n and m are interchanged, i.e. now n is one of the β's and m one of the α's. Now in the sequence $(\alpha(1), \ldots, n, \ldots, \alpha(i), \beta(1), \ldots, m, \ldots, \beta(j))$, to exchange n for m requires an odd permutation. Moreover since $m = n+1$, this interchange leaves both subsets still ordered. Thus the two dual terms in the sum $\Sigma_\alpha \pm (x_{\alpha(1)} \wedge \ldots \wedge x_{\alpha(i)} \otimes y_{\beta(1)} \wedge \ldots \wedge y_{\beta(j)})$ occur with opposite signs, and cancel. Since our function vanishes when two adjacent entries are equal, by an argument in the appendix it vanishes when any two entries are equal, hence is alternating and induces a linear map $\psi: \Lambda^s(X \oplus Y) \to \oplus_{i+j=s} (\Lambda^i(X) \otimes \Lambda^j(Y))$.
To see that φ and ψ are inverses is fairly easy. (We check it only on generators.) In one direction, $\psi(\varphi(x_1 \wedge \ldots \wedge x_i \otimes y_1 \wedge \ldots y_j)) = \psi(x_1 \wedge \ldots \wedge x_i \wedge y_1 \wedge \ldots y_j) = (x_1 \wedge \ldots \wedge x_i \otimes y_1 \wedge \ldots y_j)$. In the other, $\varphi(\psi((x_1+y_1) \wedge \ldots \wedge (x_s+y_s))) = \varphi(\Sigma_\alpha \pm (x_{\alpha(1)} \wedge \ldots \wedge x_{\alpha(i)} \otimes y_{\beta(1)} \wedge \ldots \wedge y_{\beta(j)}))$
$= \Sigma_\alpha \pm (x_{\alpha(1)} \wedge \ldots \wedge x_{\alpha(i)} \wedge y_{\beta(1)} \wedge \ldots \wedge y_{\beta(j)}) = (x_1+y_1) \wedge \ldots \wedge (x_s+y_s)$.
QED theorem.

Remark: The lemma given in the proof the previous theorem defines a "graded" algebra structure on the direct sum $\oplus_{s\geq 0} \wedge^s(M)$. If we denote this algebra by $\wedge(M)$, the isomorphism in the theorem can abbreviated as $\wedge(X\oplus Y) \cong \wedge(X)\otimes\wedge(Y)$. The algebra $\wedge(M)$ is universal in the sense that if S is any R-algebra admitting an R-module map $\varphi: M\to S$ such that $(\varphi(z))^2 = 0$ for all z in M, then there is a unique R-algebra map $\wedge(M)\to S$ which extends φ.

Exercise #195) Assume $T: V\to W$ is a linear map of finite dimensional vector spaces over a field. By rank(T) we mean the dimension of the image space $T(V)\subset W$. Prove the following:
(i) If $s >$ rank(T) then $\wedge^s(T) = 0$.
(ii) If [T] is the matrix of T for the bases $\{e_i\}$ of V and $\{f_j\}$ of W, then for the associated bases of $\wedge^s(V)$ and $\wedge^s(W)$, the entries of the matrix of $[\wedge^s(T)]$ are the s×s subdeterminants of [T].
(iii) Rank(T) = r if and only if for $s > r$ all s×s subdeterminants of [T] equal 0, but some r×r subdeterminant is non zero.

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Things to do:
Insert after the section on spectral theorems:
1) **examples of classical linear groups**, simplicity (and order) of $PSL_2(F)$, or $PSL_n(F)$ in particular existence of a **simple group of order 168**, $\cong PSL_2(\mathbb{Z}_7) \cong PSL_3(\mathbb{Z}_2)$. Note $GL_n \supset SL_n \supset \{\pm I\}$ is a normal tower where $GL_n/SL_n \cong F^*$ is abelian and $\{\pm I\} \cong \mathbb{Z}_2$. Hence one natural place to look for a new simple group is $SL_n/\{\pm I\}$

Add the following at the end:
2) **Projectives** [motivate definition via need to characterize modules for which Hom is exact], including relation to locally free ones. Example of the tangent vector fields to the sphere being projective, (topologically) locally free implies projective, projective implies flat. Give Lefschetz' proof that every continuous vector field on the sphere has a zero, and deduce that $\mathcal{V}$, the module of tangent vector fields to S, is not $\cong$ to $\mathbb{C}\times\mathbb{C}$.

3) Generalize concept of "locally free", via localization of rings, modules. Prove exactness of localization functor, Prove locally free implies projective in abstract algebraic sense. Show localization is equivalent to tensoring with the localized ring, and hence that R_S is always R-flat. Check that the nilpotents in a ring are the intersection of the prime ideals by pulling back a maximal ideal from the ring R_f localized at a non nilpotent f, to give a prime not containing f. Describe Dedekind domains as noetherian domains which are locally pid's, hence they give another generalization of pid's.

4) Injectives, existence of plenty of them, with sketch of Watts proof characterizing covariant Hom functors [this motivates definition of injectives, i.e need for something dual to free modules in sense of homology (actually dual to projectives)],

5) Inverse limits and CRT for not nec. rel prime factors, i.e. find the actual image of the natural map $R/(\prod f_\alpha) \rightarrow \prod R/(f_\alpha)$, to a product of quotients by non relatively prime factors f_α, as an inverse limit instead of a full product of the factors;

6) Inductive limits and example of power series.

Write the appendix on
7) invariance of dimension of vector spaces via Schroeder Bernstein.

8) Make an index.