ALGEBRA QUALIFYING EXAM, AUGUST 2009

(1) List all groups of order 14 up to isomorphism. Carefully justify your answer.

(2) Please show that S_4 is a solvable non-abelian group.

(3) Please identify all the conjugacy classes of elements in the group S_5. Provide an explicit representative for each class, and justify that your list is complete.

(4) Determine the group of units in each of the following rings:

$$\mathbb{Z}[i], \quad \mathbb{F}_3[x]/(x^2+1), \quad \mathbb{F}_5[x]/(x^2+1), \quad \mathbb{F}_{81}, \quad \mathbb{Z}/81\mathbb{Z}.$$

(5) Let R be any principal ideal domain. Let $n > 0$ and $A \in M_n(R)$ denote a square $(n \times n)$-matrix with coefficients in R. Consider the R-module $M := R^n/\mathrm{Im}(A)$.
   (a) Give a necessary and sufficient condition for M to be a torsion module (i.e., every non-zero element of M is torsion). Justify your answer.
   (b) Let F be a field, and let now $R := F[x]$, the ring of polynomials in one variable with coefficients in F. Given an example of an integer $n > 0$ and a $(n \times n)$-square matrix $A \in M_n(R)$ such that $M := R^n/\mathrm{Im}(A)$ is isomorphic as R-module to $R \times F$.

(6) Let R and S be two commutative rings (with multiplicative identity).
   (a) Prove that when R is a field, every non-zero ring homomorphism $\phi : R \to S$ is injective.
   (b) Does (a) still hold if we only assume that R is a domain? If yes, prove it, and if not, provide a counter-example.

(7) (a) Let K be a field. State the main theorem of Galois theory for a finite field extension L/K.
   (b) Let $\zeta_{43} := \exp(2\pi i/43)$. Describe the group of all field automorphisms $\sigma : \mathbb{Q}(\zeta_{43}) \to \mathbb{Q}(\zeta_{43})$.
   (c) How many proper subfields are there in the field $\mathbb{Q}(\zeta_{43})$?

(8) Suppose that α is a root in $\mathbb{C}$ of $P(x) := x^{17} - 2$. How many field homomorphisms are there from $\mathbb{Q}(\alpha)$ to
   (a) $\mathbb{C}$,
   (b) $\mathbb{R}$,
   (c) $\bar{\mathbb{Q}}$, an algebraic closure of $\mathbb{Q}$?
   (Justify your answers.)

(9) Let $V \neq (0)$ be a finite dimensional vector space over an algebraically closed field k. Please prove that every linear map $L : V \to V$ must have an eigenvector $v \in V$ (please provide a simple proof, without using the Cayley-Hamilton theorem, for instance.)
   Does this statement remain true if k is not algebraically closed? If yes, prove it, and if not, provide a counter-example.

(10) Let $M \in M_5(\mathbb{R})$ be a square (5×5)-matrix with real coefficients, defining a linear map $L : \mathbb{R}^5 \to \mathbb{R}^5$. Assume that when considered as an element of $M_5(\mathbb{C})$, then the scalars 0, $1+i$, and $1+2i$, are eigenvalues of M.
   (a) Show that the associated linear map L is neither injective nor surjective.
   (b) Compute the characteristic polynomial and the minimal polynomial of M.
   (c) How many fixed points can L have (that is, how many solutions to the equation $L(v) = v$ with $v \in \mathbb{R}^5$)? (Justify.)