ALGEBRA QUALIFYING EXAM, FALL 2021

Instructions: Complete all 6 problems for a total of 100 points. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts. Please fully justify all your answers.

(1) Let G be a group. An automorphism $\phi : G \to G$ is called *inner* if the automorphism is given by conjugation by a fixed group element g, i.e.,

$$\phi = \phi_g : h \mapsto ghg^{-1}.$$

(a) (5 points) Prove that the set of inner automorphisms forms a normal subgroup of the group of all automorphisms of G.
(b) (5 points) Give an example of a finite group with an automorphism which is not inner.
(c) (5 points) Denote by S_n the group of permutations of the set $\{1, \ldots, n\}$. Suppose that $g \in S_n$ sends i to g_i for $i = 1, \ldots, n$. Let (a, b) denote as usual the cycle notation for the transposition which permutes a and b. For $i \in \{1, \ldots, n-1\}$, compute $\phi_g((i, i+1))$.
(d) (5 points) Suppose that an automorphism $\phi \in \mathrm{Aut}(S_n)$ preserves cycle type, i.e., that for every element s of S_n, s and $\phi(s)$ have the same cycle type. Show that ϕ is inner. [Hint: Consider the images of generators $\phi((1,2)), \phi((2,3)), \cdots, \phi((n-1, n))$.]

(2) (15 points) Give generators and relations for the non-commutative group G of order 63 containing an element of order 9.

(3) (15 points) What is the Jordan normal form over $\mathbb{C}$ of a 7×7 matrix A which satisfies all of the following conditions:
(a) A has real coefficients,
(b) $\mathrm{rk}\, A = 5$,
(c) $\mathrm{rk}\, A^2 = 4$,
(d) $\mathrm{rk}\, A - I = 6$,
(e) $\mathrm{rk}\, A^3 - I = 4$,
(f) $\mathrm{tr}\, A = 1$?

(4) Recall that for a given positive integer n, the cyclotomic field $\mathbb{Q}(\zeta_n)$ is generated by a primitive n-th root of unity ζ_n.
(a) (5 points) What is the degree of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$?
(b) (5 points) Define what it means for a finite field extension L/K to be Galois, and prove that the cyclotomic field $\mathbb{Q}(\zeta_n)$ is Galois over $\mathbb{Q}$.
(c) (5 points) What is the Galois group of $\mathbb{Q}(\zeta_n)$ over $\mathbb{Q}$?

(d) (5 points) How many subfields of $\mathbb{Q}(\zeta_{2021})$ have degree 2 over $\mathbb{Q}$? Note that $2021 = 43 \cdot 47$.

(5) Let R be an algebra over $\mathbb{C}$ which is finite-dimensional as a $\mathbb{C}$-vector space. Recall that an ideal I of R can be considered as a $\mathbb{C}$-subvector space of R. We define the codimension of I in R to be $\dim_{\mathbb{C}} R - \dim_{\mathbb{C}} I$, the difference between the dimension of R as a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} R$, and the dimension of I as a $\mathbb{C}$-vector space, $\dim_{\mathbb{C}} I$.

(a) (5 points) Show that any maximal ideal $m \subset R$ has codimension 1.

(b) (5 points) Suppose that $\dim_{\mathbb{C}} R = 2$. Show that there exists a surjective homomorphism of $\mathbb{C}$-algebras from the polynomial ring $\mathbb{C}[t]$ to R.

(c) (5 points) Classify such algebras R for which $\dim_{\mathbb{C}} R = 2$, and list their maximal ideals.

(6) Let R be a commutative ring with unit and let M be an R-module. Define the *annihilator* of M to be

$$\mathrm{Ann}(M) := \{r \in R \,\big|\, r \cdot m = 0 \text{ for all } m \in M\}.$$

(a) (5 points) Prove that $\mathrm{Ann}(M)$ is an ideal in R.

(b) (5 points) Conversely, prove that every ideal in R is the annihilator of some R-module.

(c) (5 points) Give an example of a module M over a ring R such that each *element* $m \in M$ has a nontrivial annihilator $\mathrm{Ann}(m) := \{r \in R \,\big|\, r \cdot m = 0\}$, but $\mathrm{Ann}(M) = \{0\}$.