Complete all 6 problems. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts. Please fully justify all your answers. (Points are distributed evenly between parts unless indicated otherwise.)

1. (18 points) Suppose G is a finite group with subgroup H.

   (a) State and prove Lagrange's Theorem.

   (b) Suppose $[G : H] = 2$. Prove H is normal in G.

   (c) Prove the converse to Lagrange's Theorem does *not* hold for the group A_4. *Suggested outline*: Suppose H is a subgroup of A_4 with $|H| = 6$, and show for any $s \in A_4$, $s^2 \in H$. You may also use the fact that A_4 contains eight 3-cycles.

2. (18 points) (a) Classify all abelian groups of order 176.

   (b) Let G be a finite group, p a prime integer, and $P \subseteq G$ a p-Sylow subgroup of G. Prove that n_p, the number of p-Sylow subgroups of G, is divisible by $[G : P]$. (Hint: Consider a group action of G on the set $\mathcal{P} = \{gPg^{-1} | g \in G\}$.)

   (c) Prove that there is no simple group of order 176.

3. (18 points) Consider the following subset S of $\mathbb{C}$ given by $S = \{a + b\sqrt{2} | a, b \in \mathbb{Z}\}$.

   (a) Show S is an integral domain. (You may assume S is a subring of $\mathbb{C}$.)

   (b) Let F denote the field of fractions for S. Identify F as a subset of $\mathbb{C}$, and prove it is a field of fractions for S.

   (c) Give an explicit isomorphism between F and a quotient of the polynomial ring $\mathbb{Q}[x]$. Prove your map is an isomorphism.

4. (14 points) Suppose $F \subset L \subset K$ are field extensions, $G = G(K : F)$, and H is a subgroup of G.

   (a) Define $\mathcal{F}(H)$, the subfield of K associated to H, and $\mathcal{G}(L)$, the subgroup of G associated to L, and what it means for K to be a *Galois extension* of F.

   (b) Suppose $F = \mathbb{Q}$, ω is a primitive 5th root of 1, and $K = \mathbb{Q}(\omega)$. Is K Galois over F? Compute $[K : F]$, G, and all intermediate fields L.

5. (14 points) Consider the ring $R = \mathbb{Q}[x]$. Let $V = \mathbb{Q}^2$ written as column vectors, and define $T : V \to V$ by
$$T = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

   (a) (1 point) We may regard V as an R-module (denoted V_T) by defining $x \cdot v = T(v)$ for $v \in V$. Consider a polynomial $f(x) = a_n x^n + \cdots a_1 x + a_0 \in \mathbb{Q}[x]$. How is $f(x) \cdot v$ defined?

   (b) (2 points) Suppose R is a PID. Define the *order* of $m \in M$, denoted $|m|$, where M is an R-module.

   (c) (3 points) Compute $|v|$ and $|w|$ for the elements
$$v = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

   (d) (4 points) Show w is a cyclic vector for V_T, and compute the minimal polynomial $m_T(x) \in \mathbb{Q}[x]$.

   (f) (4 points) Identify V_T with a quotient of R as R-modules.

6. (18 points) Let
$$A = \begin{bmatrix} -1 & 2 & 3 \\ -4 & 5 & 4 \\ -2 & 1 & 4 \end{bmatrix} \in M_3(\mathbb{C}).$$

   You may use that this matrix has characteristic polynomial $-(x-3)^2(x-2)$.

   (a) Find the Jordan canonical form J of A.

   (b) Find an invertible matrix P such that $P^{-1}AP = J$. (You should not need to compute P^{-1}.)

   (c) Write down the minimal polynomial of A.