Algebra Qualifying Exam Questions

(1) Suppose that $F \subset E$ are fields such that E/F is Galois, and $|Gal(E/F)| = 14$.
  (a) Show there exists a unique intermediate field $F \subset K \subset E$, with $[K : F] = 2$.
  (b) Assume that there are at least two distinct intermediate subfields $F \subset L_i \subset E$, for $i \in \{1, 2\}$, with $[L_i : F] = 7$. Prove that G is not Abelian.

(2) Let G be a finite group, and p a prime number such that there is a normal subgroup $H \subset G$, with $|H| = p^i > 1$.
  (a) Show that H is a subgroup of any p-Sylow subgroup of G.
  (b) Show that G contains a non-identity Abelian normal subgroup of order divisible by p.

(3) Let G be a group of order 70.
  (a) Show that G is not simple.
  (b) Exhibit 3 non-isomorphic groups of order 70. Prove they are not isomorphic.

(4) Consider the polynomial $f(x) = x^7 - 3 \in \mathbb{Q}[x]$. Let $E/\mathbb{Q}$ be a splitting field of $f(x)$, and let $\alpha \in E$ be a root of $f(x)$.
  (a) Show that E contains a primitive 7th root of unity.
  (b) Show that $E \neq \mathbb{Q}(\alpha)$.

(5) Let M be a finitely generated module over a principal ideal domain R.
  (a) Let M_t be the set of torsion elements of M. Show that M_t is a submodule of M.
  (b) Show that M/M_t is torsion free.
  (c) Prove that $M \cong M_t \oplus F$ where F is a free module.

(6) Let k be a field and let the group $G = \mathrm{GL}_m(k) \times \mathrm{GL}_n(k)$ act on the set of $m \times n$ matrices, $\mathrm{M}_{m,n}(k)$ as follows:

$$(A, B) \cdot X = AXB^{-1},$$

where $(A, B) \in G$ and $X \in \mathrm{M}_{m,n}(k)$.
  (a) State what it means for a group to act on a set. Prove that the action above is indeed a group action.
  (b) Exhibit with justification a subset S of $\mathrm{M}_{m,n}(k)$ which contains precisely one and only one element of each orbit under this action.

(7) Consider the following matrix

$$A = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ -4 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

as a linear transformation from $V := \mathbb{C}^5$ to $\mathbb{C}^5$.

(a) Find the invariant factors of A.

(b) Express V in terms of a direct sum of indecomposable $\mathbb{C}[x]$-modules.

(c) Find the Jordan Canonical Form for A.

(8) Let V be a finite dimensional vector space over a field k and $T : V \to V$ be a linear transformation.

(a) Provide a definition for the minimal polynomial in $k[x]$ for T.

(b) Define the characteristic polynomial for T.

(c) Prove that the linear transformation T satisfies its characteristic polynomial (i.e., the Cayley-Hamilton Theorem)