QUALIFYING EXAMINATION IN COMPLEX ANALYSIS

August 9, 2013

$\mathbb{D}$ denotes the (open) unit disk, $\overline{\mathbb{D}}$ the closed unit disk. Provide justifications as appropriate.

1. (25 points) Let $\sqrt{z}$ denote the principal branch of the square root function and set $f(z) = \dfrac{\sqrt{z}}{(1+z^2)^2}$.
   a. Give the *singular part* of $f(z)$ at $z = i$.
   b. Use methods of complex analysis to evaluate $\displaystyle\int_0^\infty f(x)\,dx$.

   [Post-exam edit: The authors suggest that you try to use Laurent series techniques for part a. and use only your answer to part a. to do part b.]

2. (15 points) Suppose f is meromorphic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$ (away from the poles in $\mathbb{D}$) , and takes on real values on the unit circle. Prove that f is a rational function.

3. (20 points)
   a. Classify, without proof, the bijective analytic maps $\mathbb{D} \to \mathbb{D}$.
   b. Suppose $f\colon \overline{\mathbb{D}} \to \mathbb{C}$ is continuous, $f\colon \mathbb{D} \to \mathbb{D}$ is analytic, and $|f(z)| = 1$ when $|z| = 1$. Prove that if f is not a constant function, then f takes on every value in $\mathbb{D}$.

4. (20 points)
   a. Suppose $f = u + iv$ is holomorphic at a. Explain carefully the relation between the complex derivative $f'(a)$ and the Jacobian matrix at a of f viewed as a map from $\mathbb{R}^2$ to $\mathbb{R}^2$ (denoted $J_f(a)$). In particular, how is $\det J_f(a)$ related to $f'(a)$?
   b. Consider $X = \{(z,w) \in \mathbb{C}^2 : z^3 + 3z^2w + w^3 = 5\}$. Consider $(z_0, w_0) = (-2, 1) \in X$. Prove that there are neighborhoods U and $V \subset \mathbb{C}$ of z_0 and w_0, respectively, so that $X \cap (U \times V)$ can be represented as the graph $w = g(z)$ for some *holomorphic* function $g\colon U \to V$. (Hint: Start with the implicit function theorem and a map from $\mathbb{R}^4$ to $\mathbb{R}^2$. But do *not* write it out in real coordinates.)

5. (20 points) Suppose $\mathcal{F}$ is a family of holomorphic functions on $\mathbb{D}$ with the property that $\displaystyle\int_{\mathbb{D}} |f(z)|\,dx\,dy \le \pi$ for all $f \in \mathcal{F}$.
   a. Show that for every $f \in \mathcal{F}$ we have $|f(0)| \le 1$. (Hint: Use polar coordinates.) One can show similarly—but you need not do so—that if $0 < R \le 1$ and $|z| \le 1 - R$, then $|f(z)| \le 1/R^2$.
   b. Prove that for any compact subset $K \subset \mathbb{D}$ there is a constant $M > 0$ such that $|f'(z)| \le M$ for all $z \in K$ and for all $f \in \mathcal{F}$.