Complex Analysis Qualifying Examination

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All problems are of equal weight. Do the easier ones first. Please arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.

Notation: $\mathbb{D}, \mathbb{H}$ denote respectively the unit disc centered at origin and the upper half plane.

1. Without using Cauchy's theorem or residue theory, work out (a)-(c) below.

   (a) Evaluate the integrals $\displaystyle\int_\gamma z^n dz$ for all integers n. Here γ is any circle centered at the origin with the positive (counterclockwise) orientation.

   (b) Same question as before, but with γ any circle not containing the origin.

   (c) Show that if $|a| < r < |b|$, then

   $$\int_\gamma \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b},$$

   where γ denotes the circle centered at the origin, of radius r, with the positive orientation

2. Show that if $a > 0$, then

   $$\int_0^\infty \frac{\log x}{x^2 + a^2} = \frac{\pi}{2a} \log a$$

3. Let $P(z)$ be a polynomial with distinct roots $z_1, z_2, \ldots z_m$. Show that there exist c_k such that

   $$\frac{1}{P(z)} = \sum_{k=1}^{m} \frac{c_k}{z - z_k}.$$

4. Suppose f is entire and there exist $A, R > 0$ and natural number N with $|f(z)| \geq A|z|^N$ for $|z| \geq R$. Show that (i) f is a polynomial and (ii) the degree of f is at least N.

5. (1) Show that if f is analytic in an open set containing the disc $|z - a| \leq R$, then

   $$|f(a)|^2 \leq \frac{1}{\pi R^2} \int_0^{2\pi} \int_0^R |f(a + re^{i\theta})|^2 r dr d\theta$$

   (2) Let Ω be a region and $M > 0$ a fixed positive constant. Let $\mathcal{F}$ be the family of all analytic functions f on Ω such that $\displaystyle\iint_\Omega |f(z)|^2 dxdy \leq M$. Show that $\mathcal{F}$ is uniformly bounded on compact subsets of Ω.

6. Assume $f(z)$ is analytic in $|z| < 1$ and $|f(z)| < 1$. Show that

$$\left|\frac{f(z) - f(z_0)}{1 - \overline{f(z_0)} f(z)}\right| \leq \left|\frac{z - z_0}{1 - \bar{z}_0 z}\right|$$

7. Let $G = \{z : \ |z-1| < \sqrt{2}, \ |z+1| < \sqrt{2}\} \backslash [0, i)$. Find a bijective conformal map from G to the upper half plane.