Epistolary Math Tournament - Fall MMXXI

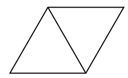
University of Georgia

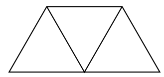
Friday October 1st

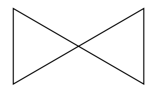
Set 1 - Solution

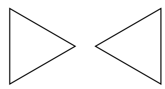
Problem 1

An icosahedron is a solid whose twenty faces are all equilateral triangles, five of which meet at each of its vertices. How many of each of the following shapes can you find on its surface?

(a) (2pts) Two triangles sharing a single edge. 

(b) (2pts) Three triangles as shown here. 

(c) (3pts) Two triangles sharing no edges and only one common vertex. 

(d) (3pts) Two completely disjoint triangles. 

Problem 2

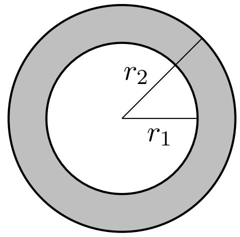
(a) (4pts) Find all polynomials with integer coefficients of the form

$$p(x) = x^2 + p(a)x + a.$$

(b) (6pts) List all polynomials $p(x)$ with integer coefficients such that

$$p(x) = x^2 + p(p(a))x + a.$$

Problem 3



Set-up. Consider the annulus A pictured above. Define an *xy-path* between two points P_1 and P_2 in A to be a sequence of line segments connecting P_1 and P_2 which satisfy the following properties:

1. The path stays completely in A (touching the boundary is allowed).
2. Each segment of the path must be parallel to the x-axis or the y-axis.
3. The number of segments is minimal.

Examples:

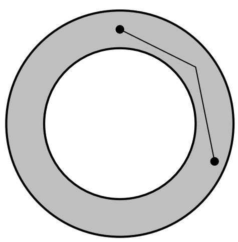


Fig. 1

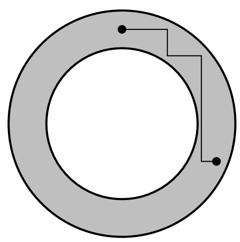


Fig. 2

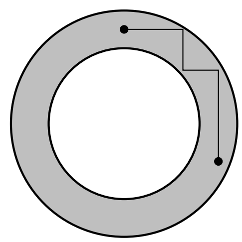


Fig. 3

Figure 1 is not an xy-path as its segments are not parallel to the axes.
Figure 2 is not an xy-path as it has 5 segments but a 4 segment path is possible.
Figure 3 is an xy-path.

(a) (3pts) Given annulus A with inner radius $r_1 = 2$ and outer radius $r_2 = 3$, what is the number of segments on an xy-path between the uppermost point on A and the lowermost point on A?

(b) (7pts) Given annulus A with inner radius $r_1 = 20$ and outer radius $r_2 = 21$, what is the number of segments on an xy-path between the uppermost point on A and the lowermost point on A?

Solution 1

It is known that an icosahedron has 12 vertices, 30 edges, and 20 faces. This can be derived from the given information:

Let V be the number of vertices, E the number of edges, and F the number of faces of the icosahedron. Now, from the given, each vertex is shared by five edges, so naively counting suggests that $E = 5V$; however, this is not correct, since each edge has two vertices, this way of counting counts each edge exactly twice. Thus, in fact, $E = 5V/2$, or $V = 2E/5$. Similarly, each face has three edges; however, each edge lies on exactly two vertices, and counting and correcting for double-counting gives us $E = 3F/2$, or $F = 2E/3$. By Euler's polyhedron formula, $V - E + F = 2$, or $2E/5 - E + 2E/3 = E/15 = 2$. Thus $E = 30, V = 12, F = 20$.

(a) Notice that each edge of the icosahedron, corresponding to the shared edge of the two faces, determines exactly one such shape. Thus there are exactly 30 such shapes.

(b) Consider the central face of each shape; there are exactly 20 ways to choose such, and three ways to choose two out of three adjacent faces to comprise the rest of the shape. Thus there are exactly $20 \times 3 = 60$ such shapes.

(c) Note that each such shape corresponds to exactly one vertex. Thus, we need to count the number of ways to choose two disjoint triangles meeting at each vertex. At each vertex, there are five triangles, and thus $\binom{5}{2} = 10$ ways to choose two of them; however, exactly five of each will give two consecutive triangles (as in (a)) sharing a common edge. Thus, there are five ways to choose two triangles at each vertex, and so we have $5 \times 12 = 60$ such shapes.

(d) There are exactly $\binom{20}{2} = 190$ ways to choose two triangles. However, if they are not disjoint, they share either exactly one edge or exactly one vertex. From (a), the former happens exactly 30 times; from (c), the latter happens exactly 60 times. Thus, there are exactly $190 - 30 - 60 = 100$ such pairs of disjoint triangles.

Solution 2

(a) Evaluating at a we get

$$p(a) = a^2 + p(a)a + a.$$

which implies that $a \neq 1$ as well as

$$p(a) = \frac{a^2 + a}{1 - a} = -a - 2 + \frac{2}{1 - a}.$$

Since $p(a)$ is an integer, $1 - a$ is either ± 2 or ± 1, i.e a priori $a \in \{-1, 0, 2, 3\}$. We thus have four cases; $p(x)$ is one of the following polynomials:

$$x^2 - 1, x^2, x^2 - 6x + 2, x^2 - 6x + 3.$$

(b) First note that if $a = 0$, then

$$p(x) = x^2 + p(p(0))x + 0 = x^2 + p(0)x = x^2.$$

For the following we freely assume $a \neq 0$ so we can safely divide by a.

Observe that $p(a)$ is divisible by a. In particular $p(a) = an$ with $n := a + p(p(a)) + 1$. This means that $p(p(a)) = p(na)$ is also divisible by a, so let $p(p(a)) = ar$ and the above definition of n becomes $n = a + ar + 1$.

Now we can write $p(na) = p(p(a))$ in two ways, first from the result of plugging in na into $p(x)$ and second by rearranging the equation $n = a + p(p(a)) + 1$ to get $p(p(a)) = n - a - 1$. Hence

$$\begin{aligned}
&n^2a^2 + (n - a - 1)na + a = n - a - 1 \\
\implies &n^2a^2 - na^2 + (n - 1)na - (n - 1) = -2a \\
\implies &(n - 1)na^2 + (n - 1)na - (n - 1) = -2a \\
\implies &(n - 1)(n(a^2 + a) - 1) = -2a \\
\implies &(a + ar)(n(a^2 + a) - 1) = -2a \\
\implies &(1 + r)(n(a^2 + a) - 1) = -2
\end{aligned}$$

Hence $(1 + r) \mid 2$, so $r \in \{-3, -2, 0, 1\}$. This gives 4 cases. Rewriting using $n = (a + ar + 1)$, we can search see if there are a values which work for a given r value in the equation

$$(a + ar + 1)(a^2 + a) = 1 - \frac{2}{1 + r}.$$

Case 1: $r = -3$. The given equation becomes

$$(-2a + 1)(a^2 + a) = 2.$$

This has no solutions as $-2a+1$ must be an odd divisor of 2, so either $a=0$ or $a=1$, neither of which work.

Case 2: $r=-2$. The given equation becomes

$$(-a+1)(a^2+a)=3.$$

This has no solutions because a^2+a is even for any integer.

Case 3: $r=0$. The given equation becomes

$$(a+1)(a^2+a)=-1.$$

This too has no solutions because a^2+a is even for any integer.

Case 4: $r=1$. The given equation becomes

$$(2a+1)(a^2+a)=0.$$

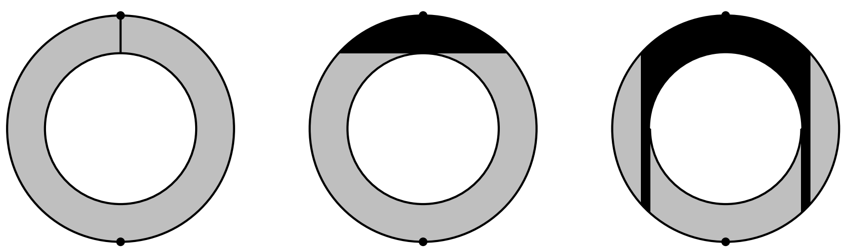
This has the integral solutions $a=0$ and $a=-1$. We have dealt with the $a=0$ case already, and indeed $a=-1$ with $p(p(a))=ar=-1(1)=-1$ gives rise to the other valid solution

$$p(x)=x^2-x-1.$$

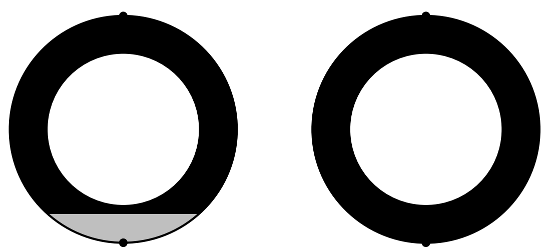
The only polynomials which work are therefore $p(x)=x^2$ and $p(x)=x^2-x-1$.

Solution 3

(a) We will give a pictorial explanation here and leave the computational details to the solution of the second part of this problem. We consider sequentially which points in the annulus we can reach with xy-paths of at most length n starting with the uppermost point.



$n = 1$ $\quad$ $n = 2$ $\quad$ $n = 3$



$n = 4$ $\quad$ $n = 5$

We can reach the lowestmost point in 5 segments but not 4, so the answer is 5.

(b) By symmetry, it suffices to find the number of segments needed to reach some point on the midline of the annulus starting from the uppermost point. We can then reflect the path vertically to get an xy-path from top to bottom. In our effort to get to the mid-line, we may as well choose each segment to have maximum possible length. Depending on the radii, the number of segments needed will change, and the boundary case occurs when we are just able to "turn the corner," i.e. when the midline point the path described above reaches is a point also on the inner circle.

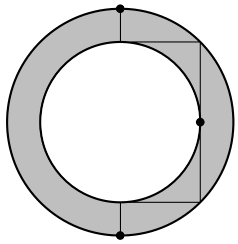


Fig. 1

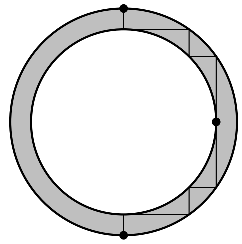


Fig. 2

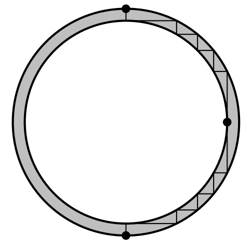
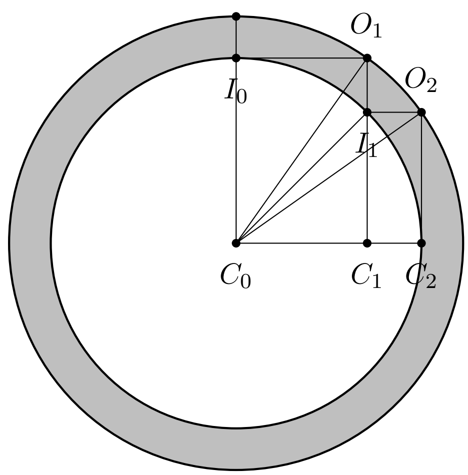


Fig. 3

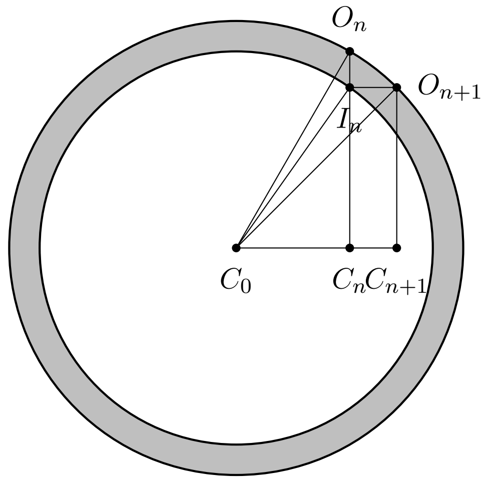
Notice that in each of the above figures if we increase the inner radius at all then the number of segments on the xy-path must increase. As an example, we can calculate the corresponding condition on the radii in figure 2 then in generality.



Using right triangle $\triangle C_0O_1C_1$, we compute that $O_1 = \left(\sqrt{r_2^2 - r_1^2}, r_1\right)$. Using right triangle $\triangle C_0I_1C_1$, we compute that $I_1 = \left(\sqrt{r_2^2 - r_1^2}, \sqrt{2r_1^2 - r_2^2}\right)$. Using right triangle $\triangle C_0O_2C_2$, we compute that $O_2 = \left(\sqrt{2r_2^2 - 2r_1^2}, \sqrt{2r_1^2 - r_2^2}\right)$. However, the x-coordinate of O_2 should be r_1, so we get

$$\sqrt{2r_2^2 - 2r_1^2} = r_1 \implies \frac{r_1}{r_2} = \sqrt{\frac{2}{3}}.$$

Now for the general case:



In general, if $O_n = \left(\sqrt{nr_2^2 - nr_1^2}, \sqrt{nr_1^2 - (n-1)r_2^2}\right)$, then by right triangle $\triangle C_0 I_n C_n$, we compute that $I_n = \left(\sqrt{nr_2^2 - nr_1^2}, \sqrt{(n+1)r_1^2 - nr_2^2}\right)$. Finally, using right triangle right triangle $\triangle C_0 O_{n+1} C_{n+1}$, we compute that

$$O_{n+1} = \left(\sqrt{(n+1)r_2^2 - (n+1)r_1^2}, \sqrt{(n+1)r_1^2 - nr_2^2}\right).$$

For some O_n, the x-coordinate will need to be the same as r_1, so in general we have

$$\sqrt{nr_2^2 - nr_1^2} = r_1 \implies \frac{r_1}{r_2} = \sqrt{\frac{n}{n+1}}.$$

Checking against the figures from before, we can see that if O_n is the point on the outer circle with x-coordinate r_1, then an xy-path of length $4n+1$ is barely possible (i.e. becomes impossible if we increase r_1 by any amount). Hence if $\sqrt{\frac{n-1}{n}} < \frac{r_1}{r_2} \leq \sqrt{\frac{n}{n+1}}$, then the number of segments on an xy-path is $4n+1$. Note that $\left(\frac{20}{21}\right)^2 = \frac{400}{441}$ is a little less that $\frac{400}{440} = \frac{10}{11}$. Therefore we conclude that if $r_1 = 20$ and $r_2 = 21$, then the number of segments on an xy-path between the uppermost point and lowermost point is $4(10) + 1 = 41$.

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