TIDBITS OF GEOMETRY THROUGH THE AGES

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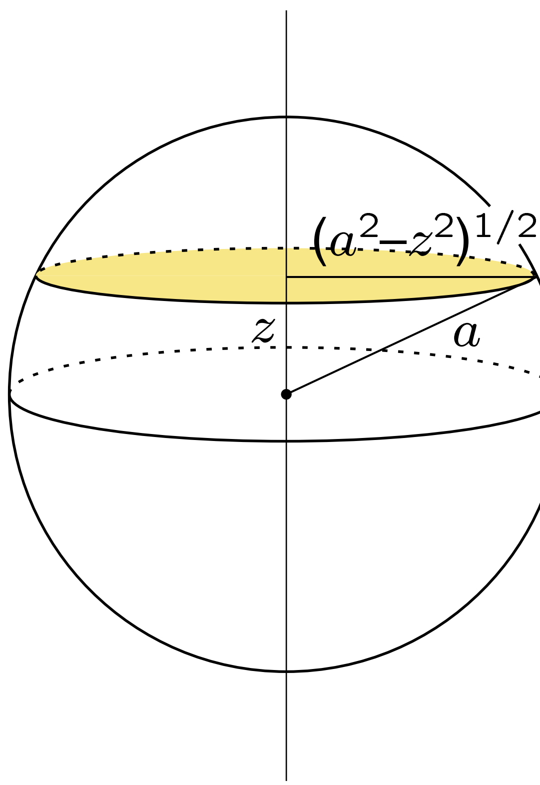
1. *Euclidean* geometry and Archimedes.

Problem (a) The volume of the sphere

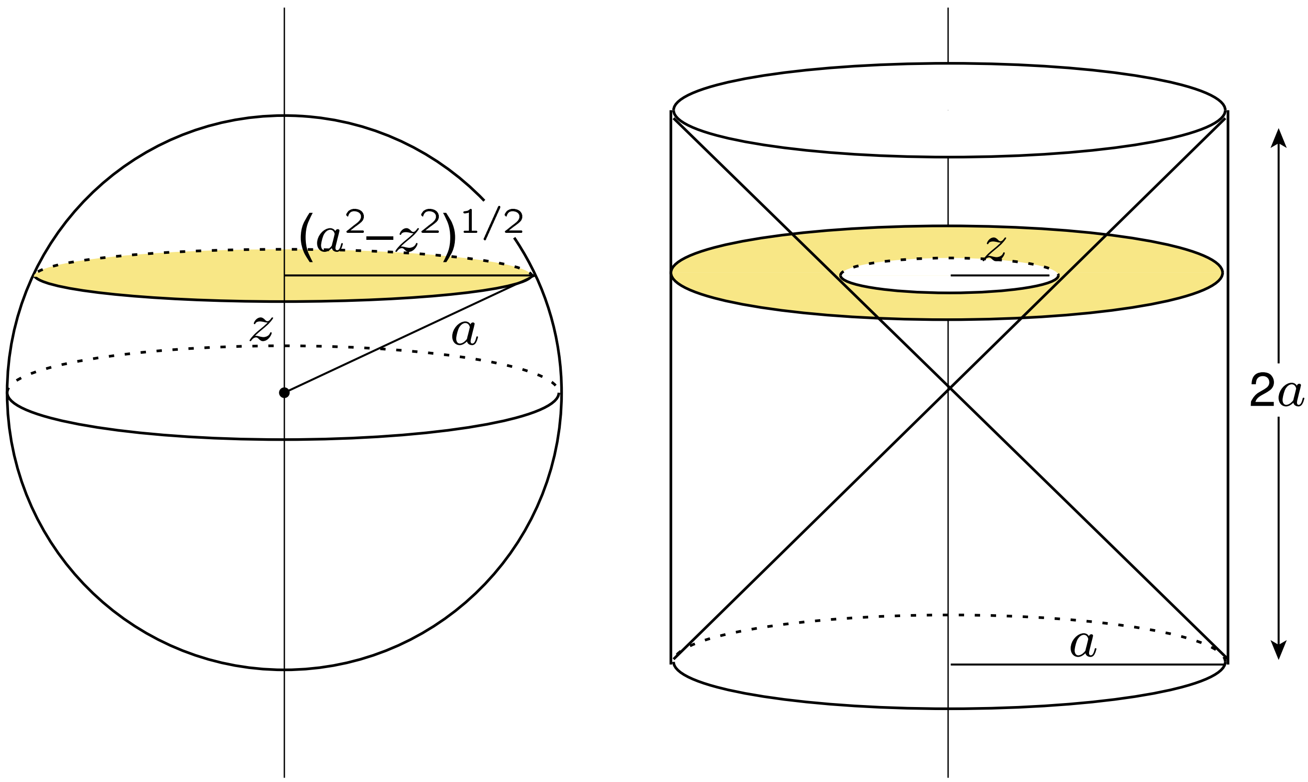
$A(z) =$ area of cross-section at height z



$$
\begin{aligned}
\text{Volume} &= \int_{-a}^{a} A(z)\,dz = \int_{-a}^{a} \pi(a^2 - z^2)\,dz \\
&= \pi\left(a^2 z - \frac{1}{3}z^3\right)\Big|_{-a}^{a} = \frac{4}{3}\pi a^3 .
\end{aligned}
$$

How did Archimedes deduce the formula for the volume of a ball?

Hint: Cavalieri's principle.



We get the same cross-sectional area at height z if we construct a cylinder of base a and height $2a$ and cut out a (double-) cone of the same dimensions.

Then the volume of the cylinder less the volume of the cone is

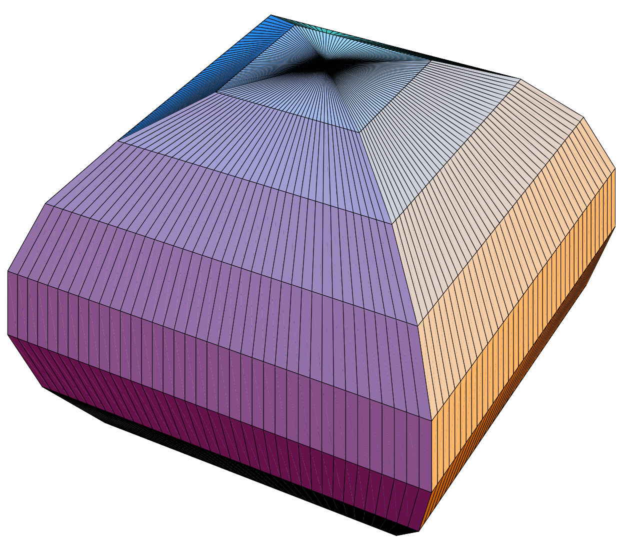
$$\pi a^2(2a) - 2 \cdot \frac{1}{3}\pi a^2(a) = \frac{4}{3}\pi a^3,$$

as we expected.

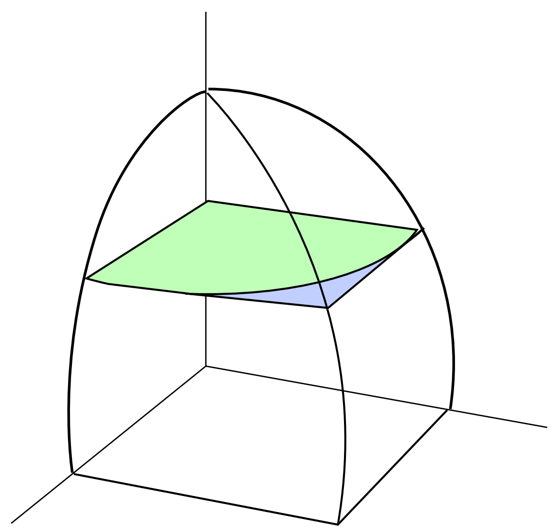
Problem (b) Another standard integral calculus problem: The volume of the bicylinder.

What is the volume of the region bounded by the two orthogonal cylinders

$$x^2 + z^2 = a^2 \qquad \text{and} \qquad y^2 + z^2 = a^2\,?$$



This is again an easy application of Cavalieri's principle: comparing to a sphere, the cross-section at height z is a square circumscribing the corresponding cross-section of a sphere of radius a.



Thus, the cross-sectional areas are in a ratio of $4/\pi$ and the volume of the bicylinder is

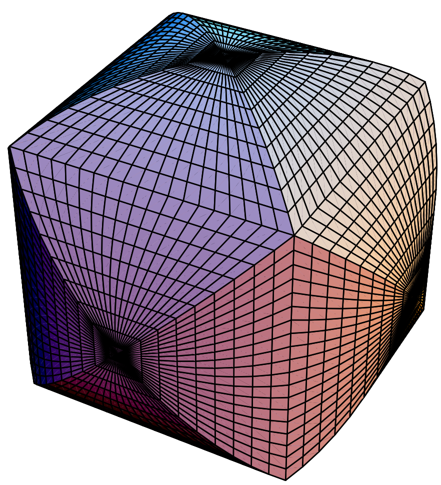
$$\frac{4}{\pi} \cdot \frac{4}{3}\pi a^3 = \frac{16}{3}a^3.$$

(This is approx. 1.27 times the volume of the ball of radius a.)

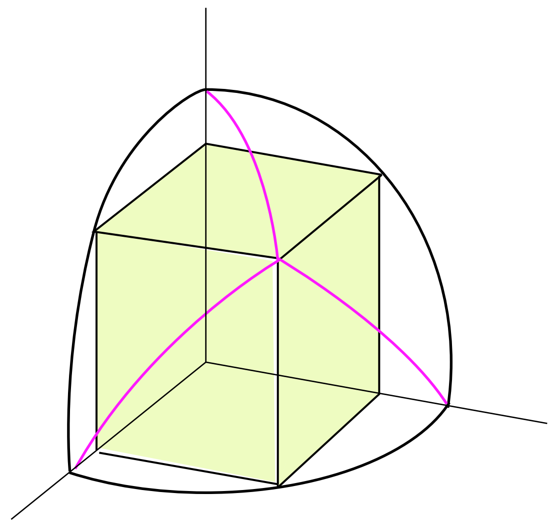
Problem (c) My favorite extra-credit multiple integration problem: The volume of the tricylinder.

What is the volume of the region bounded by the *three* orthogonal cylinders

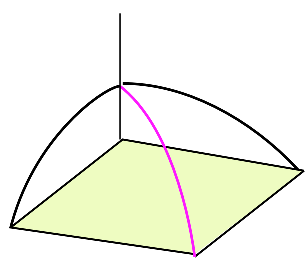
$x^2+y^2=a^2$, $x^2+z^2=a^2$, and $y^2+z^2=a^2$?



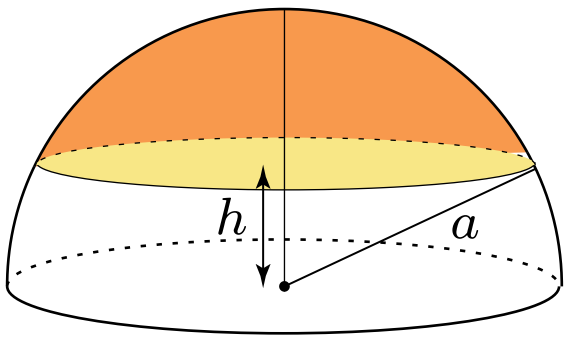
Now, how would the Greeks have found the volume?

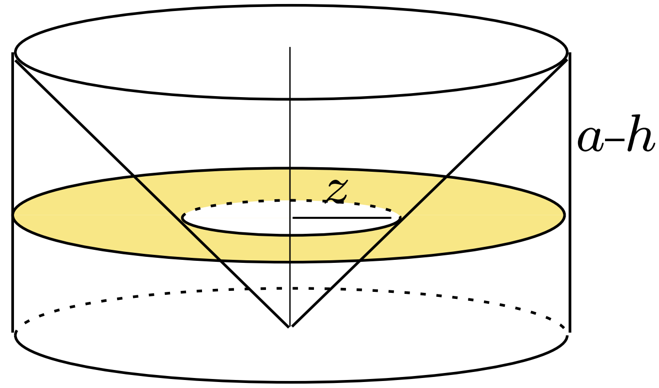


Consider the portion in the first octant. We have a cube and three “caps” that are parts of a truncated bicylinder.



By comparing each of these to the corresponding segment of a ball, we can use Archimedes' approach to find the volume.





Volume of truncated ball

$$= \pi a^2(a-h) - \frac{\pi}{3}(a^3 - h^3)$$
$$= \frac{\pi}{3}(a-h)^2(2a+h)$$

With $h = a/\sqrt{2}$, this is

$$= \frac{\pi}{3}a^3 \cdot \frac{4\sqrt{2}-5}{2\sqrt{2}}$$

Therefore, volume of truncated bicylinder

$$= 6 \cdot \frac{4}{\pi} \cdot \frac{\pi}{3}a^3 \cdot \frac{4\sqrt{2}-5}{2\sqrt{2}}$$
$$= \frac{8}{2\sqrt{2}}(4\sqrt{2}-5)a^3$$

Adding the volume of the inner cube, the volume of the tricylinder is equal to

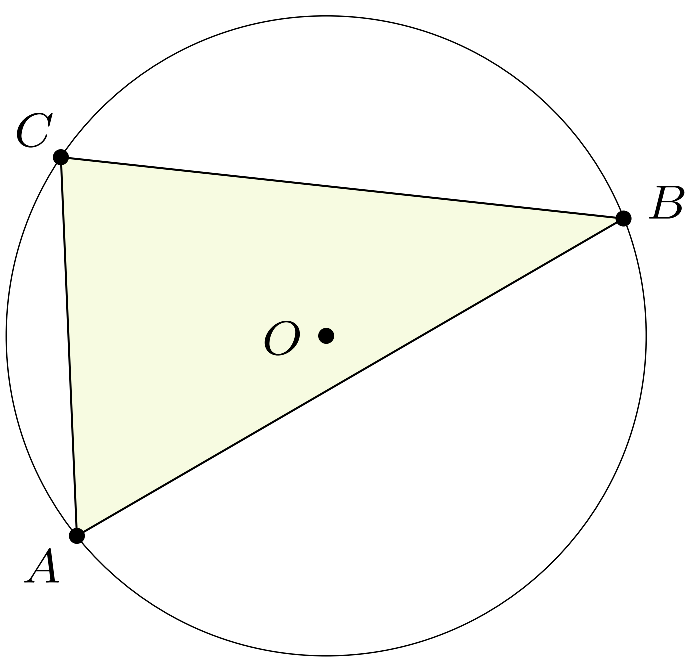
$$(a\sqrt{2})^3 + 2\sqrt{2}(4\sqrt{2} - 5)a^3 = \boxed{8(2 - \sqrt{2})a^3}\,.$$

(This is approximately 1.19 times the volume of the ball of radius a.)

Challenge: Do this by calculus, by either single- or double-integration!

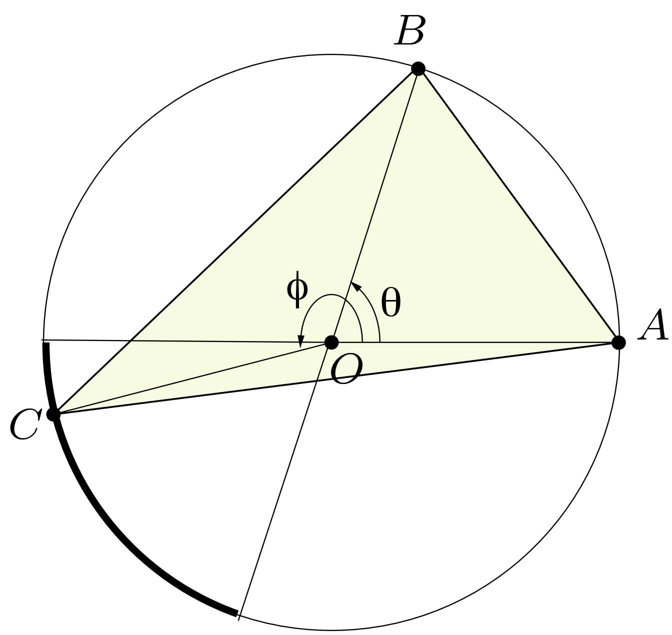
2. *Affine* geometry. A recent Putnam exam contained a three-dimensional version of the following problem:

Given three points chosen at random on the unit circle, what is the probability that the center of the circle lies in the interior of the triangle they form?



Approach (a) Straightforward calculus solution.

Without loss of generality, put A at $(1,0)$, set $B = (\cos\theta, \sin\theta)$ and $C = (\cos\phi, \sin\phi)$.



We see O lies inside $\triangle ABC$ precisely when

$$\phi \in (\pi, \pi + \theta)\,.$$

That is, assuming C is chosen at random, the probability that O is inside the triangle is $\theta/2\pi$.

We now average over all possible positions of B:

$$\text{Prob } (O \text{ inside } \triangle ABC) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\theta}{2\pi}\, d\theta = \frac{1}{2}.$$

(Try solving the 3-D problem in an analogous fashion! Good luck!)

Approach (b) Try linear algebra (actually "affine linear algebra"). We say the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^2$ are *affinely independent* if

If $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$ and $x + y + z = 0$, then $x = y = z = 0$.

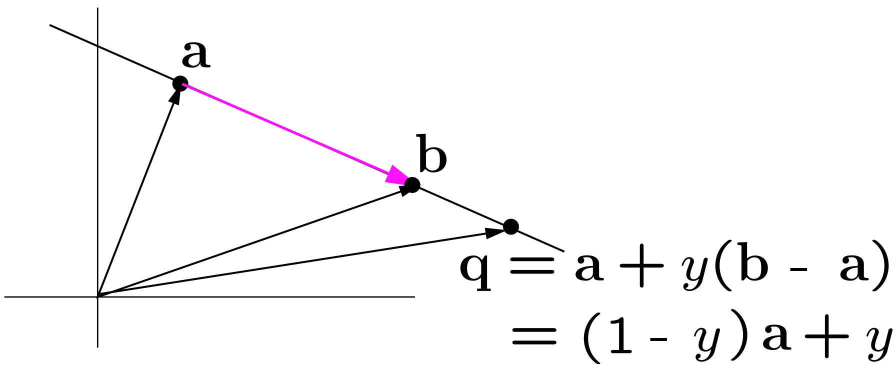
It is an easy exercise to check that

$\mathbf{a}, \mathbf{b}, \mathbf{c}$ are *affinely* independent

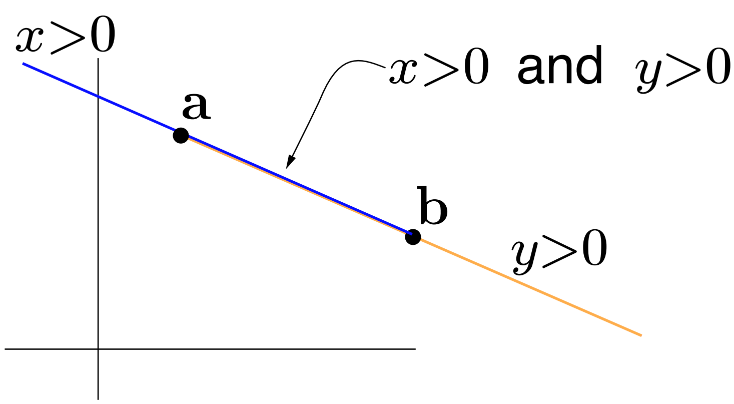
$\iff$

$(\mathbf{b} - \mathbf{a}), (\mathbf{c} - \mathbf{a})$ are *linearly* independent.

When $\mathbf{a} \neq \mathbf{b}$, every point on the line through $\mathbf{a}$ and $\mathbf{b}$ can be expressed as an *affine linear combination* $x\mathbf{a} + y\mathbf{b}$, where $x + y = 1$.



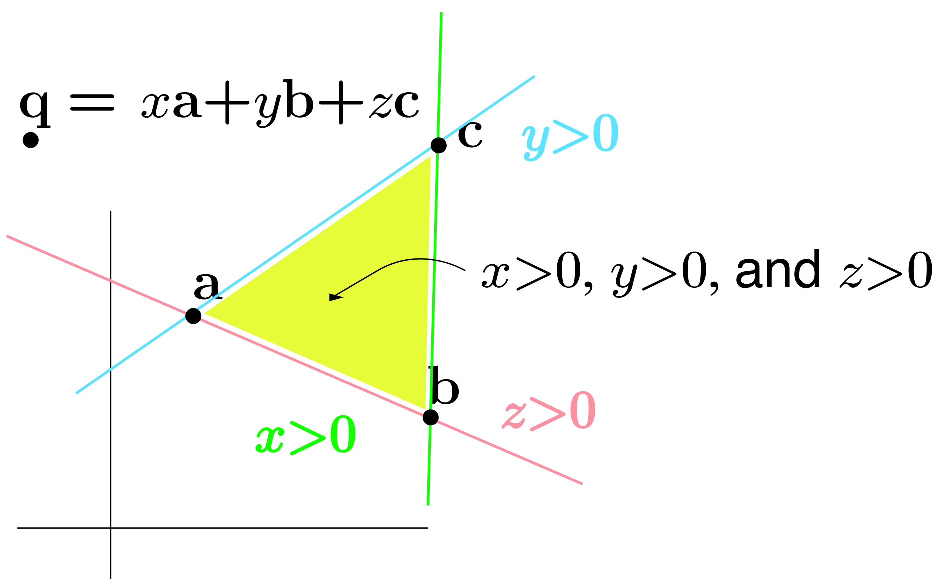
Moreover, we see that when $x > 0$ the point is on the $\mathbf{a}$-side of $\mathbf{b}$ and when $y > 0$ the point is on the $\mathbf{b}$-side of $\mathbf{a}$. In particular, points where $x, y \geq 0$ correspond to the line segment $\overline{\mathbf{ab}}$.



Likewise, when $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are affinely independent, every point in $\mathbb{R}^2$ can be written as an affine linear combination

$$x\mathbf{a} + y\mathbf{b} + z\mathbf{c}\,, \quad \text{where} \quad x + y + z = 1\,.$$

And a point is in the interior of the triangle with vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ if and only if $x > 0$, $y > 0$, and $z > 0$.



Now we're back to our question: When is O in the interior of $\triangle ABC$?

Precisely when we can write

$$\mathbf{0} = x\mathbf{a}+y\mathbf{b}+z\mathbf{c} \quad \text{with} \quad x > 0, y > 0, \text{ and } z > 0\,.$$

Since we have, correspondingly,

$$\mathbf{0} = x(-\mathbf{a}) + y(-\mathbf{b}) + z(-\mathbf{c})\,,$$

this means that O will also be in the interior of the triangle with vertices at $-A$, $-B$, and $-C$. On the other hand, we have

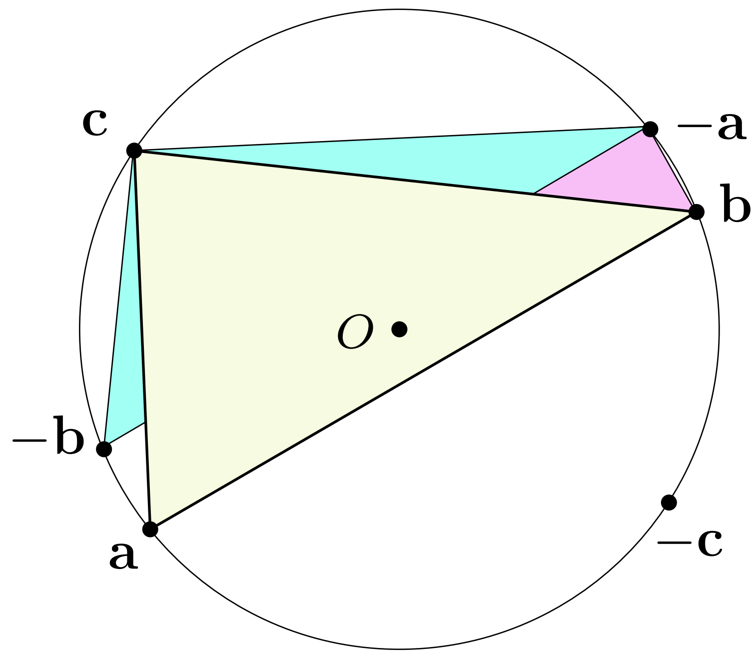
$$\begin{aligned} \mathbf{0} &= (-x)(-\mathbf{a}) + y\mathbf{b} + z\mathbf{c} \\ \mathbf{0} &= x\mathbf{a} + (-y)(-\mathbf{b}) + z\mathbf{c} \\ \mathbf{0} &= x\mathbf{a} + y\mathbf{b} + (-z)(-\mathbf{c})\,, \end{aligned}$$

and so on, so that O will be in the exterior of the remaining six triangles with vertices $\pm A$, $\pm B$, and $\pm C$.

In summary, given three points A, B, and C chosen at random on the unit circle, of the $\mathbf{8}$ possible triangles with vertices

$$\pm A\,, \quad \pm B\,, \quad \pm C\,,$$

there will be precisely $\mathbf{2}$ with O in the interior.



Since the antipodal map is an isometry of the circle, the probability that a point lies in an interval is equal to the probability that its antipodal point lies in the antipodal interval. Thus, choosing three points at random on the unit circle, the probability that the origin is in the interior of the triangle they determine is

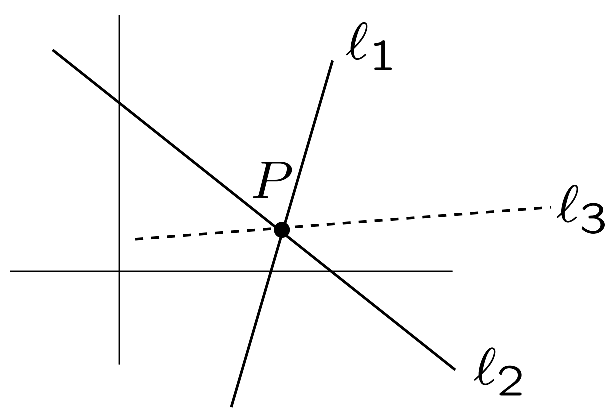
$$\frac{2}{8} = \frac{1}{4}.$$

3. *Projective geometry.* Here is a question from classical linear algebra.

Given three distinct lines in the plane in parametric form:

$$\begin{aligned} \ell_1: \quad \mathbf{x} &= \mathbf{a}_1 + s\mathbf{v}_1 \\ \ell_2: \quad \mathbf{x} &= \mathbf{a}_2 + t\mathbf{v}_2 \\ \ell_3: \quad \mathbf{x} &= \mathbf{a}_3 + u\mathbf{v}_3 \,, \end{aligned}$$

what is a criterion for the three lines to be *concurrent*, i.e., to have a point in common?



Approach (a) First, we solve for the point of intersection, P, of ℓ_1 and ℓ_2: since

$$s\mathbf{v}_1 - t\mathbf{v}_2 = \mathbf{a}_2 - \mathbf{a}_1 \,,$$

Cramer's rule tells us that

$$s = \frac{D(\mathbf{a}_2 - \mathbf{a}_1, \mathbf{v}_2)}{D(\mathbf{v}_1, \mathbf{v}_2)}\,,$$

where $D(\mathbf{x}, \mathbf{y})$ denotes the determinant of the 2×2 matrix with column vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$.

In order for ℓ_3 to pass through P, we must have

$$s\mathbf{v}_1 - u\mathbf{v}_3 = \mathbf{a}_3 - \mathbf{a}_1\,,$$

so Cramer's rule gives, once again,

$$s = \frac{D(\mathbf{a}_3 - \mathbf{a}_1, \mathbf{v}_3)}{D(\mathbf{v}_1, \mathbf{v}_3)}\,.$$

Setting these two expressions for s equal, we have

$$\begin{aligned} &D(\mathbf{a}_3, \mathbf{v}_3)D(\mathbf{v}_1, \mathbf{v}_2) + D(\mathbf{a}_2, \mathbf{v}_2)D(\mathbf{v}_3, \mathbf{v}_1) \\ &\quad = \; D(\mathbf{a}_1, \mathbf{v}_3)D(\mathbf{v}_1, \mathbf{v}_2) + D(\mathbf{a}_1, \mathbf{v}_2)D(\mathbf{v}_3, \mathbf{v}_1)\,. \end{aligned}$$

But we can rewrite the right-hand side as

$$D\left(\mathbf{a}_1, D(\mathbf{v}_1, \mathbf{v}_2)\mathbf{v}_3 + D(\mathbf{v}_3, \mathbf{v}_1)\mathbf{v}_2\right)\,,$$

which in turn is equal to

$$-D(\mathbf{a}_1, D(\mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1) = -D(\mathbf{a}_1, \mathbf{v}_1)D(\mathbf{v}_2, \mathbf{v}_3)\,.$$

To see this, we need the

Lemma. For any three vectors in $\mathbb{R}^3$, we have

$$D(\mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1 + D(\mathbf{v}_3, \mathbf{v}_1)\mathbf{v}_2 + D(\mathbf{v}_1, \mathbf{v}_2)\mathbf{v}_3 = \mathbf{0}.$$

Finally, we have derived the criterion for concurrence:

$$\begin{aligned} D(\mathbf{a}_3, \mathbf{v}_3)D(\mathbf{v}_1, \mathbf{v}_2) + D(\mathbf{a}_2, \mathbf{v}_2)D(\mathbf{v}_3, \mathbf{v}_1)& \\ +D(\mathbf{a}_1, \mathbf{v}_1)D(\mathbf{v}_2, \mathbf{v}_3) &= 0 \end{aligned}$$

Proof of Lemma.

$D(\mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1 + D(\mathbf{v}_3, \mathbf{v}_1)\mathbf{v}_2 + D(\mathbf{v}_1, \mathbf{v}_2)\mathbf{v}_3 = \mathbf{0}.$

Possible proof: use Cramer's Rule again. But, back to affine linear combinations: if

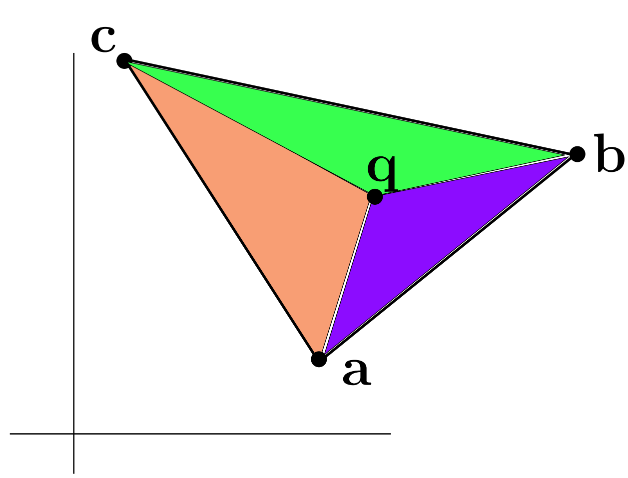
$$\mathbf{q} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}\,, \quad \text{with} \quad x + y + z = 1\,,$$

then

$$x = \frac{\text{signed area } \triangle\mathbf{qbc}}{\text{signed area } \triangle\mathbf{abc}}\,, \quad \text{etc.},$$

so, letting $\mathbf{q} = \mathbf{0}$, we get

$$x = \frac{\text{signed area } \triangle\mathbf{0bc}}{\text{signed area } \triangle\mathbf{abc}} = \frac{\frac{1}{2}D(\mathbf{b}, \mathbf{c})}{\ldots}\,.$$



$\mathbf{q} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$

Approach (b) Suppose instead we are given cartesian equations of the lines:

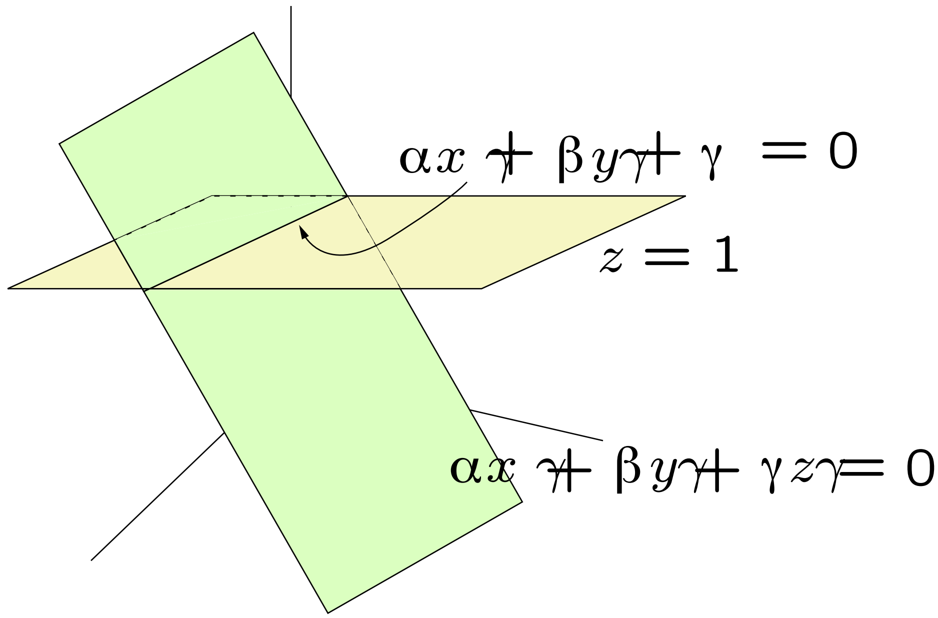
$$\begin{aligned}
\ell_1 : \quad \alpha_1 x + \beta_1 y + \gamma_1 &= 0 \\
\ell_2 : \quad \alpha_2 x + \beta_2 y + \gamma_2 &= 0 \\
\ell_3 : \quad \alpha_3 x + \beta_3 y + \gamma_3 &= 0 \,.
\end{aligned}$$

Now it is much easier to give the condition for concurrence.

Consider the corresponding planes through the origin in $\mathbb{R}^3$:

$$\begin{aligned}
\mathcal{P}_1 : \quad \alpha_1 x + \beta_1 y + \gamma_1 z &= 0 \\
\mathcal{P}_2 : \quad \alpha_2 x + \beta_2 y + \gamma_2 z &= 0 \\
\mathcal{P}_3 : \quad \alpha_3 x + \beta_3 y + \gamma_3 z &= 0 \,.
\end{aligned}$$

Their respective intersections with the plane $z = 1$ recover the lines ℓ_i.



The planes intersect in at least a line (and so the corresponding lines will intersect in at least a point) provided

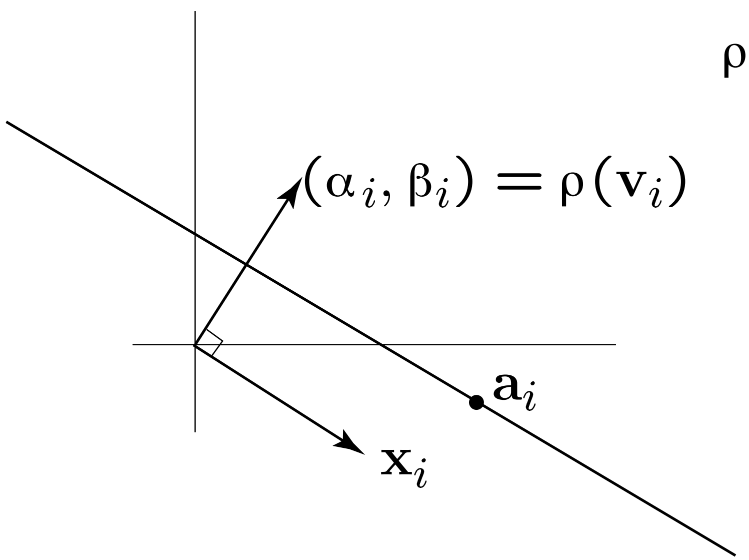
$$\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} = 0\,.$$

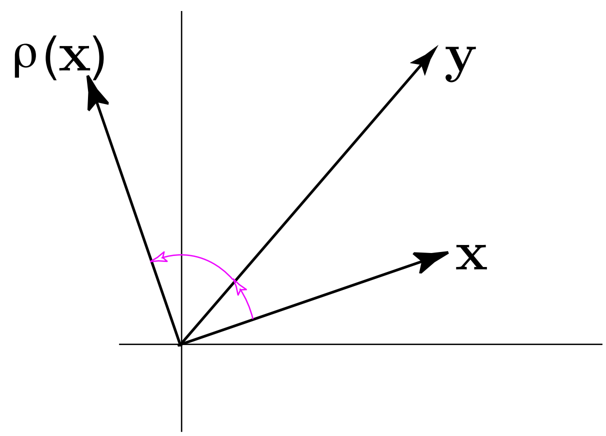
Expanding in cofactors along the third column, we can recover our earlier equation with a bit of thought. We have

$$(\alpha_i, \beta_i) = \rho(\mathbf{v}_i)\,,$$

where $\rho(x, y) = (-y, x)$ gives rotation of the plane $\pi/2$ counterclockwise, and

$$\gamma_i = -(\alpha_i, \beta_i) \cdot \mathbf{a}_i = -\rho(\mathbf{v}_i) \cdot \mathbf{a}_i = D(\mathbf{a}_i, \mathbf{v}_i)\,.$$





The last ingredient is merely that, for example,

$$\begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} = D(\rho(\mathbf{v}_1), \rho(\mathbf{v}_2)) = D(\mathbf{v}_1, \mathbf{v}_2)$$

since ρ preserves signed area. So we get

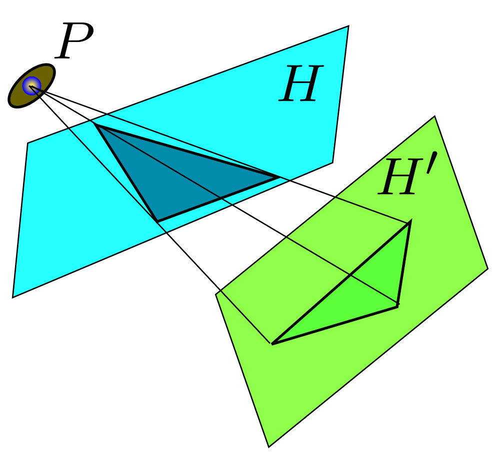
$$\gamma_1 D(\mathbf{v}_2, \mathbf{v}_3) + \gamma_2 D(\mathbf{v}_3, \mathbf{v}_1) + \gamma_3 D(\mathbf{v}_1, \mathbf{v}_2) = 0\,,$$

just as before.

What underlies this result is *projective duality*: lines in a projective plane are concurrent if and only if the corresponding points in the dual projective plane are collinear.

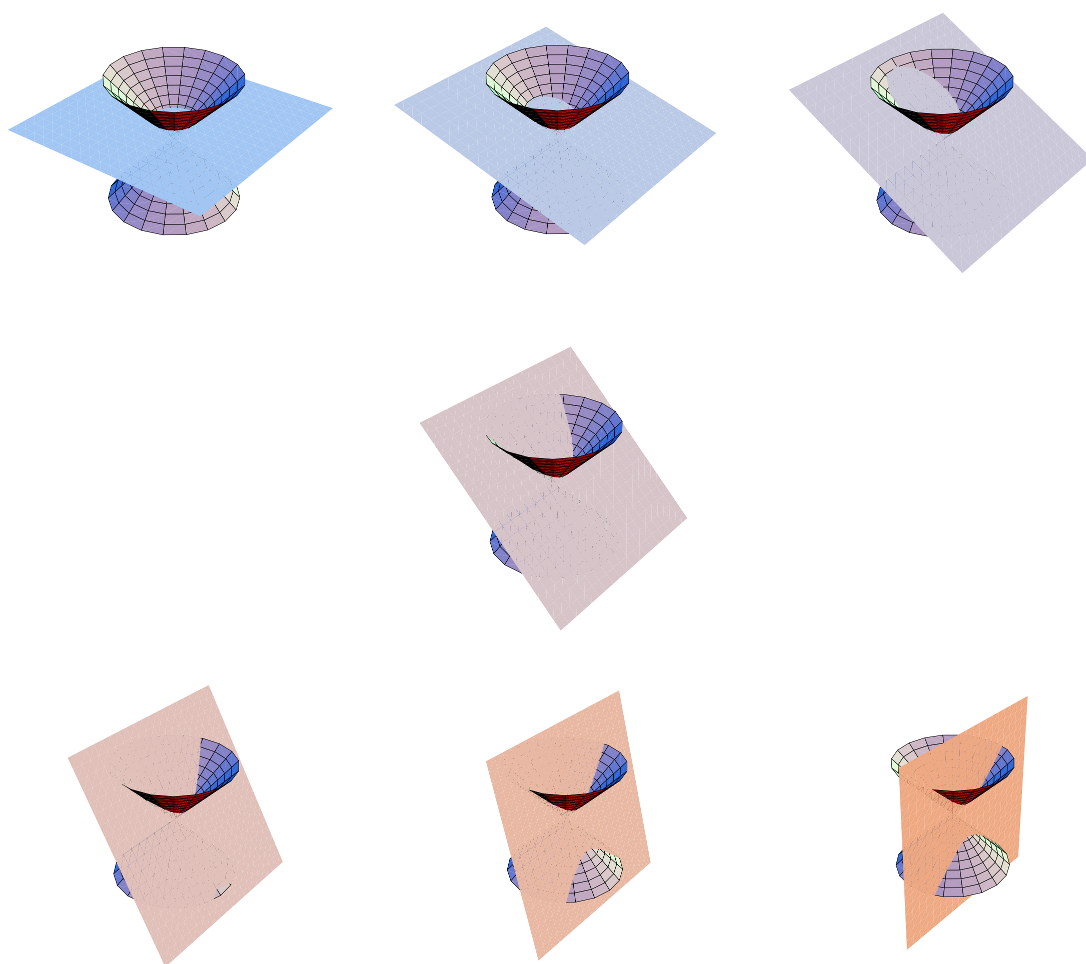
4. A bit more *projective* geometry, leading to *algebraic* geometry.

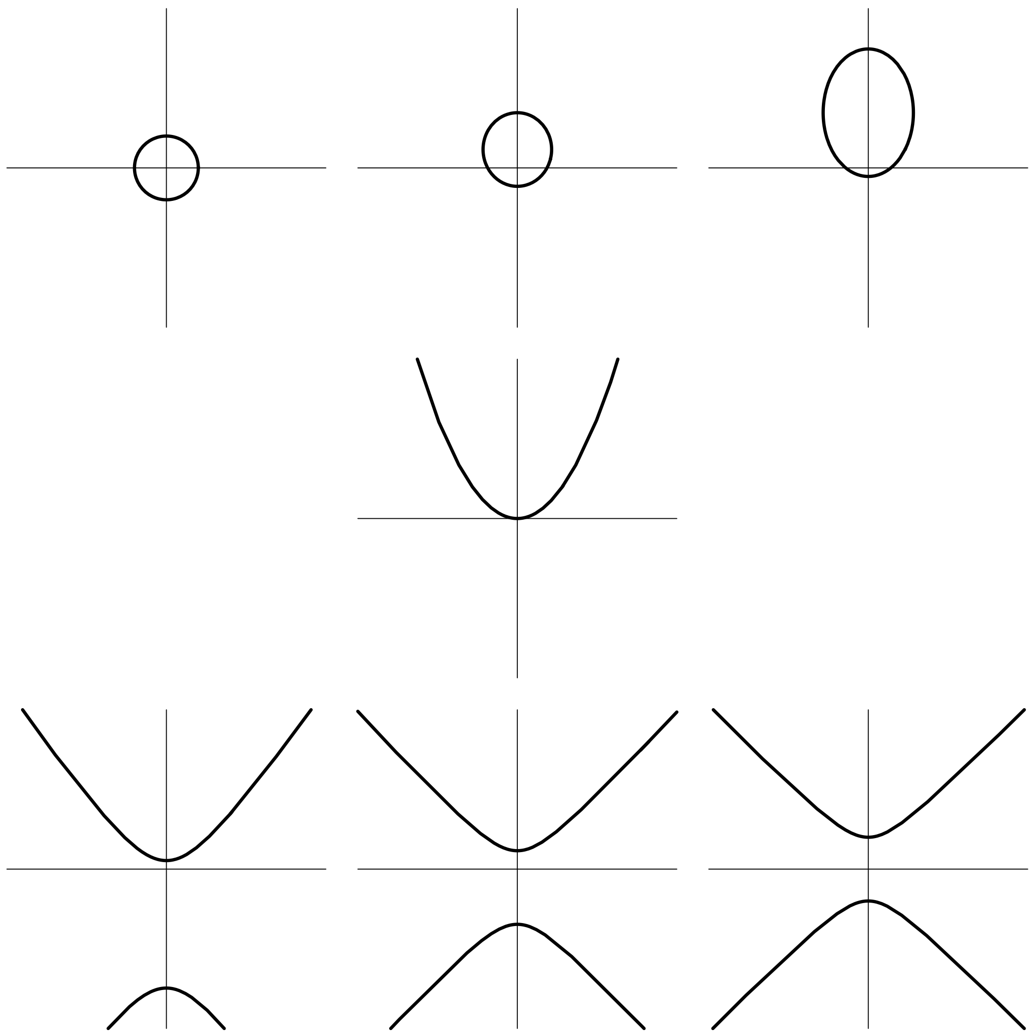
Naive definition: Two plane sets C and C' are *projectively equivalent* if we can choose two planes H, H' and a point P so that, viewing from P, $C' \subset H'$ is the image of $C \subset H$ subtended in H'.



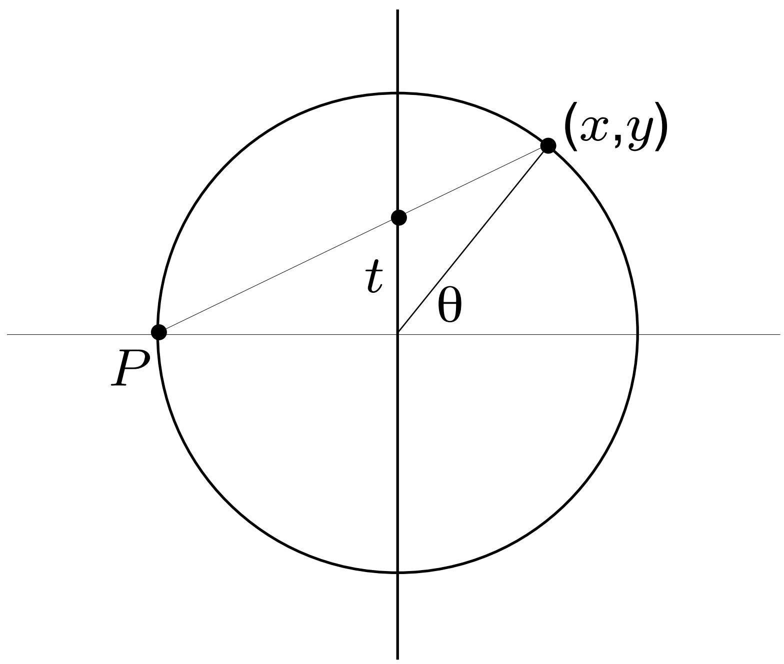
For example, standard conics—circles, ellipses, parabolas, and hyperbolas—are projectively equivalent.

The circle we get by slicing a cone with a horizontal plane above the vertex is seen on various viewing planes as ...





For something different: If we view a circle from a point P *on* the circle, it turns into a line (with a point "at infinity").



Projecting from the point $P = (-1, 0)$ we obtain

$$t = \frac{x+1}{y}$$

and so we obtain

$$(ty-1)^2 + y^2 = 1 \Longrightarrow (t^2+1)y^2 - 2ty = 0\,.$$

Thus,

$$x = \frac{1 - t^2}{1 + t^2} \qquad \text{and} \qquad y = \frac{2t}{1 + t^2}\,.$$

This is the famous *rational* parametrization of the circle. Here are two important applications:

Application (a) "Rationalizing substitution"

$$\int \frac{d\theta}{3\cos\theta + 4\sin\theta} =?$$

Substituting $x = \cos\theta$, $y = \sin\theta$, and noticing that $t = \tan(\theta/2)$, we obtain $d\theta = 2\,dt/(1+t^2)$, and the integral becomes

$$\int \frac{2\,dt}{3 + 8t - 3t^2}\,.$$

Since $-3t^2 + 8t + 3 = -(3t + 1)(t - 3)$, this is easily evaluated.

Application (b) This gives us an explicit formula for all Pythagorean triples. Triples of *integers* (X, Y, Z) with $X^2+Y^2 = Z^2$ correspond to pairs of *rational numbers*

$$x = \frac{X}{Z} \qquad \text{and} \qquad y = \frac{Y}{Z}$$

with $x^2 + y^2 = 1$. Any such rational point corresponds to a rational number t.

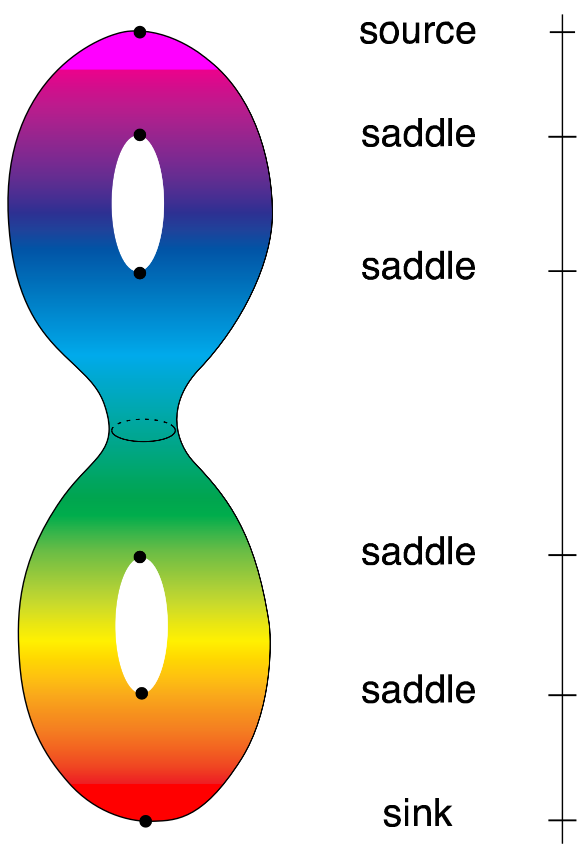
Obvious query: Can we similarly generate integral solutions of $X^n + Y^n = Z^n$? Can we parametrize the algebraic curves $x^n + y^n = 1$ by rational functions when $n \geq 3$?

The theory of algebraic curves tells us that a nonsingular ("smooth") plane curve of degree n has *genus* $g = (n-1)(n-2)/2$, and a curve can be parametrized by rational functions only when $g = 0$. (Technical point: All of this is most easily done working over $\mathbb{C}$.)

5. *Integral* and *differential* geometry (via topology).

Poincaré, Hopf, and Morse proved (in generalizing to higher dimensions) that the *Euler characteristic* of a surface S is given by

$$\chi(S) = \#(\text{sources}) - \#(\text{saddles}) + \#(\text{sinks})$$

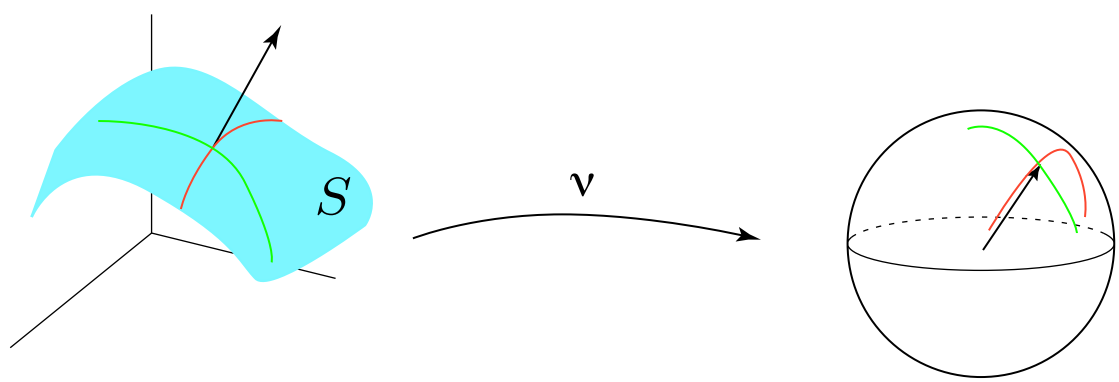


(When S has genus g, we get $\chi(S) = 2 - 2g$.)

The original Gauss-Bonnet Theorem related this to the geometry of a closed surface:

$$\boxed{\chi(S) = \frac{1}{2\pi}\int_S K\,dA.}$$

Here K is *Gaussian curvature*, a measure of how the Gauss map ν of S distorts area.

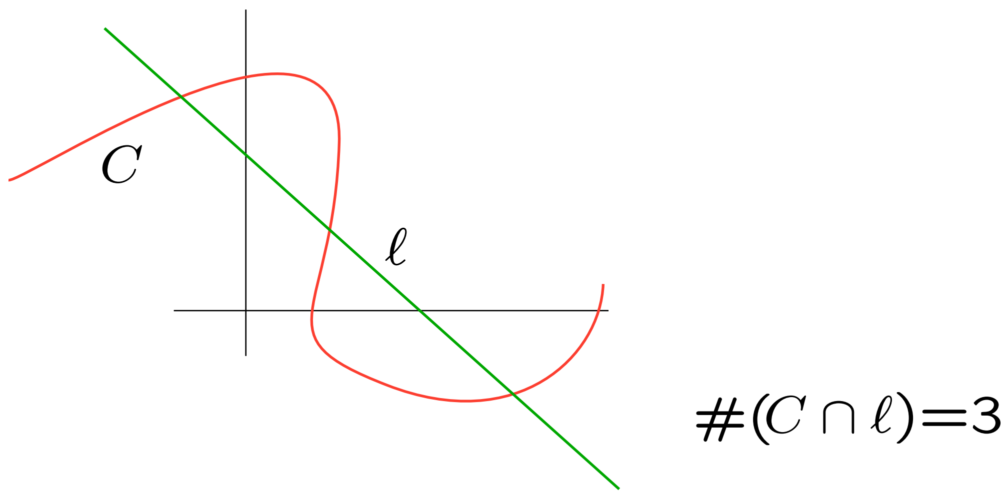


What do we do if we have just a piece of surface?

A warm-up result: For a (piece of) curve $C \subset \mathbb{R}^2$, we have

Crofton's formula:

$$\text{length}(C) = \frac{1}{2}\int \#(C \cap \ell)\, d\ell$$



(Here $d\ell$ is an obvious measure on the space of affine lines in the plane, invariant under the group of motions.) This baby result was really the beginnings of *geometric probability*, leading, for example, to a probabilistic "computation" of π. (Buffon needle problem)

There are analogous results in higher dimensions (e.g., Cauchy).

We have analogously: for any (piece of) surface $S \subset \mathbb{R}^3$:

Local Gauss-Bonnet formula:

$$\int_S K\, dA = \int \mu(L)\, dL\,,$$

where L is a line through the origin, dL is the invariant measure on the space of lines through the origin, and

$$\mu(L) = \#(\text{sources}) - \#(\text{saddles}) + \#(\text{sinks})$$

when we project onto L.

Similarly, if we just count how many critical points there are for projection onto L (without regard to their nature), we get

$$\int_S |K|\, dA = \int |\mu|(L)\, dL\,.$$

Total curvature has been an interesting invariant for studying knottedness (Fary-Milnor, Chern-Lashof).

Shameless self-promotion:

T. Shifrin, *Abstract Algebra: A Geometric Approach*, Prentice Hall, 1996.

T. Shifrin and Malcolm Adams, *Linear Algebra: A Geometric Approach*, W. H. Freeman, 2001.

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