Probability Theory, Ph.D Qualifying, Spring 2020

Completely solve any five problems.

1. If $\{A_n\}$ are events satisfying $\lim_{n\to\infty} P(A_n) = 0$ and $\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$, show that

$$P(A_n, \text{ infinitely often}) = 0.$$

2. (a) If $\{X_n,\ n \geq 1\}$ are r.v.s with $\sup_{n\geq 1} E|X_n|^\beta < \infty$ for some $\beta > 0$, then $\{|X_n|^\alpha,\ n \geq 1\}$ is uniformly integrable for $0 < \alpha < \beta$.

   (b) Prove that for any r.v. X

$$E|X| = \int_0^\infty P(|X| \geq t)dt.$$

3. Prove for nondegenerate i.i.d. r.v.s $\{X_n\}$ that $P(X_n \text{ converges}) = 0$.

4. Let $X_1, \ldots, X_n$ be a random sample from a distribution with $E(X_i) = 0$ and $Var(X_i) = 1$. Show that as $n \to \infty$,

$$Y_n = \frac{\sqrt{n} S_n}{\sum_{i=1}^n X_i^2} \to N(0,1),$$

   and

$$Z_n = \frac{S_n}{\sqrt{\sum_{i=1}^n X_i^2}} \to N(0,1),$$

   where $S_n = X_1 + \cdots + X_n$.

5. If the independent L^1 random variables $X_1, \ldots, X_n, \ldots$ satisfy the condition

$$Var(X_i) \leq c < \infty,\ i = 1, 2, \ldots,$$

   then the SLLN holds, i.e.,

$$\frac{1}{n}\sum_{i=1}^n (X_i - EX_i) \to 0, \text{a.s.}$$

6. If $\{X_n\}$ are iid $\mathcal{L}^1$ random variables, then $\sum_{n=1}^\infty \frac{X_n}{n}$ converges a.s. if either (i) X_1 is symmetric or (ii) $E|X_1| \log^+ |X_1| < \infty$ and $EX_1 = 0$.

7. Let $\{\xi_n,\ n \geq 1\}$ be independent random variables such that for some $0 < p < 1$, $P(\xi_n = 1) = p$, $P(\xi_n = -1) = 1 - p = q$. For $n \geq 1$, let $\eta_n = \sum_{k=1}^n \xi_k$ and $\zeta_n = (q/p)^{\eta_n}$. Show that $\{\zeta_n,\ n \geq 1\}$ is a martingale.