Probability Theory, Ph.D Qualifying, Fall 2020

Completely solve any five problems.

1. Show that for any two random variables X and Y with $Var(X) < \infty$,
$$Var(X) = E[Var(X|Y)] + Var[E(X|Y)].$$

2. Suppose that X and Y are independent random variables with the same exponential density
$$f(x) = \theta e^{-\theta x}, \ x > 0.$$
Show that the sum $X + Y$ and the ratio X/Y are independent.

3. Let X be a real valued random variable with mean value μ and variance σ^2. Show that for $x > 0$,
$$P(X - \mu \geq x) \ \leq \ \frac{\sigma^2}{\sigma^2 + x^2}.$$

4. Let $X, \{X_n, \ n \geq 1\}, \{Y^{(k)}, \ k \geq 1\}, \{Y_n^{(k)}, \ n \geq 1, \ k \geq 1\}$, be real random variables such that $Y_n^{(k)} \to Y^{(k)}$ in distribution as $n \to \infty$, for fixed k, and that $Y^{(k)} \to X$ in distribution as $k \to \infty$. Show that $X_n \to X$ in distribution if
$$\lim_{k\to\infty} \ \limsup_{n\to\infty} \ E[\min(|Y_n^{(k)} - X_n|, 1)] = 0.$$

5. Let $(X_n)_{n\geq 1}$ be a sequence of independent, real valued random variables with mean $\mu(X_n) = 0$. Let $S_n = X_1 + \ldots + X_n$. Show that for all $a > 0$:
$$P\{\max_{k\leq n} |S_k| > a\} \ \leq \ \frac{Var(S_n)}{a^2}.$$

6. Let $X_1, X_2, \ldots$ be a sequence of independent r.v.s with $EX_i = 0$. Let $S_n = X_1 + X_2 + \cdots + X_n$ and $\mathcal{F}_n = \sigma\{X_1, \ldots, X_n\}$. Show that $\phi(S_n)$ is an $\mathcal{F}_n$–submartingale for any convex ϕ provided that $E|\phi(S_n)| < \infty$ for all n.