ANALYSIS QUALIFYING EXAM - JANUARY 2021

Problem 1. Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $E_n \in \mathcal{M}$ be a measurable set for $n \geq 1$. Let $f_n = \chi_{E_n}$, the indicator function of the set E_n. Prove that

(a) $f_n \to 1$ uniformly, if and only if there exists $N \in \mathbb{N}$ such that $E_n = X$ for all $n \geq N$.

(b) $f_n(x) \to 1$ for almost every x, if and only if

$$\mu\Big(\bigcap_{n\geq 0}\bigcup_{k\geq n}(X\backslash E_k)\Big) = 0.$$

Problem 2. Calculate the limit:

$$L := \lim_{n\to\infty}\int_0^n \frac{\cos(x/n)}{x^2+\cos(x/n)}\,dx.$$

Justify each step of your calculations!

Problem 3. Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Let $\{f_n\}_{n=1}^{\infty} \subseteq L^1(X,\mu)$ and $f \in L^1(X,\mu)$ such that $f_n(x) \to f(x)$ as $n \to \infty$ for almost every $x \in X$.

Prove that for every $\varepsilon > 0$ there exist $M > 0$, and a set $E \subseteq X$, such that $\mu(E) \leq \varepsilon$ and $|f_n(x)| \leq M$ for all $x \in X\backslash E$ and all $n \in \mathbb{N}$.

Problem 4. Let f and g be Lebesgue integrable on $\mathbb{R}$.
Let $g_n(x) = g(x-n)$. Prove that

$$\lim_{n\to\infty}\|f+g_n\|_1 = \|f\|_1 + \|g\|_1.$$

Problem 5. Let $f_n \in L^2[0,1]$ for $n \in \mathbb{N}$. Assume that

(a) $\|f_n\|_2 \leq n^{-51/100}$, for all $n \in \mathbb{N}$, and

(b) $\hat{f}_n$ is supported in the interval $[2^n, 2^{n+1}]$, that is

$\hat{f}_n(k) = \int_0^1 f_n(x)\, e^{-2\pi i\, kx}\, dx = 0$, for $k \notin [2^n, 2^{n+1}]$.

Prove that $\sum_{n=1}^{\infty} f_n$ converges in the Hilbert space $L^2([0,1])$.

(Hint: Plancherel's identity may be helpful.)

Problem 6. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a measurable function, and for $x \in \mathbb{R}$ define the set

$$E_x := \{y \in \mathbb{R} : \ m(\{z \in \mathbb{R} : \ f(x,z) = f(x,y)\}) > 0\}.$$

Show that

$$E := \bigcup_x \{x\} \times E_x$$

is a measurable subset of $\mathbb{R} \times \mathbb{R}$.

(Hint: consider the measurable function $h(x,y,z) := f(x,y) - f(x,z)$.)