By providing my signature below, I acknowledge that I abide by the University's academic honesty policy. This is my work, and I did not get any help from anyone else:

Name (sign): ____________ Name (print): Solutions

Student Number: ____________

Instructor's Name: ____________ Class Time: ____________

Problem Number	Points Possible	Points Earned
1	24	
2	15	
3	30	
4	15	
5	20	
6	22	
7	10	
8	10	
9	10	
10	12	
11	15	
12	20	
13	15	
14	15	
15	15	
Total:	248	

- If you need extra space use the last page. *Do not tear off the last page!*
- Please show your work. **An unjustified answer may receive little or no credit.**
- If you make use of a theorem to justify a conclusion, then state the theorem used by name.
- Your work must be **neat**. If I can't read it (or can't find it), I can't grade it.
- The total number of possible points that is assigned for each problem is shown here. The number of points for each subproblem is shown within the exam.
- You may not use a cell phone or smart watch. Please turn off your cell phones, and store them and your smart watches away from your desk or table.
- You are only allowed to use a **TI-30XS Multiview** calculator. No other calculators are permitted, and sharing of calculators is not permitted.
- You do not have to use a calculator; answers containing symbolic expressions such as $\cos(\pi/3)$ and $\ln(e^4)$ are acceptable. Include an exact answer for each problem.

1. Determine the following limits; briefly explain your thinking on each one. If you apply L'Hopital's rule, indicate where you have applied it and why you can apply it. If your final answer is "does not exist," ∞, or $-\infty$, briefly explain your answer. (You will not receive full credit for a "does not exist" answer if the answer is ∞ or $-\infty$.)
Please circle your final answer.

(a) [6 pts] $\lim_{x\to 0} \frac{e^{2x}-2}{e^x+1} = \frac{1-2}{1+1} = -\frac{1}{2}$

(b) [6 pts] $\lim_{x\to 3^-} \frac{x^2+7}{x-3} = -\infty$

$x^2+7 \to 16$ as $x \to 3^-$
$x-3 \to 0$ and is negative as $x \to 3^-$

(c) [6 pts] $\lim_{x\to \pi/2} \frac{\cos(x)}{x-\frac{\pi}{2}} \overset{LH}{=} \lim_{x\to \pi/2} \frac{-\sin(x)}{1} = -\sin(\pi/2) = -1$

IF $\frac{0}{0}$

Method 1:

(d) [6 pts] $\lim_{x\to\infty} \frac{3x^2(x-1)}{x^2-2} \frac{(1/x^2)}{(1/x^2)} = \lim_{x\to\infty} \frac{3(x-1)}{1-2/x^2} = \infty$

IF $\frac{\infty}{\infty}$

$3(x-1) \to \infty$, $1-2/x^2 \to 1$ as $x \to \infty$

OR Method 2: $\overset{LH}{=} \lim_{x\to\infty} \frac{3x^2(1)+(x-1)(6x)}{2x} \overset{LH}{=} \lim_{x\to\infty} \frac{6x+(x-1)(6)+(6x)(1)}{2} = \infty$

IF $\frac{\infty}{\infty}$

$6x+6(x-1)+6x \to \infty$ as $x \to \infty$

OR Method 3: $= \lim_{x\to\infty} \frac{3x^3-3x^2}{2x} = \lim_{x\to\infty} \frac{9x^2-6x}{2} = \lim_{x\to\infty} \frac{x(9x-6)}{2} = \infty$

IF $\frac{\infty}{\infty}$

$x(9x-6) \to \infty$ as $x \to \infty$

2. (a) [5 pts] State the limit definition of the derivative of $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \underline{\text{OR}} \quad f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

(b) [10 pts] Use the **limit definition of the derivative** to determine the derivative of $f(x) = \sqrt{x-3}$. No points will be awarded for the application of differentiation rules (and L'Hopital's rule is not allowed).

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{(x+h)-3} - \sqrt{x-3}}{h} \cdot \frac{\sqrt{x+h-3} + \sqrt{x-3}}{\sqrt{x+h-3} + \sqrt{x-3}}$$

$$= \lim_{h \to 0} \frac{(x+h-3) - (x-3)}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{x+h-3} + \sqrt{x-3}\right)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h-3} + \sqrt{x-3}}$$

$$= \frac{1}{2\sqrt{x-3}}$$

—OR—

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$$= \lim_{x \to a} \frac{\sqrt{x-3} - \sqrt{a-3}}{x-a} \cdot \frac{\sqrt{x-3} + \sqrt{a-3}}{\sqrt{x-3} + \sqrt{a-3}}$$

$$= \lim_{x \to a} \frac{(x-3) - (a-3)}{(x-a)\left(\sqrt{x-3} + \sqrt{a-3}\right)}$$

$$= \lim_{x \to a} \frac{x-a}{(x-a)\left(\sqrt{x-3} + \sqrt{a-3}\right)}$$

$$= \lim_{x \to a} \frac{1}{\sqrt{x-3} + \sqrt{a-3}}$$

$$= \frac{1}{2\sqrt{a-3}}$$

3. Determine the first derivative of each of the following functions. Remember to use correct notation to write your final answer. **Please circle your final answer.**

(a) [6 pts] $f(x) = \dfrac{x^3}{4} - \dfrac{2}{x^3} + \pi^2 = \frac{1}{4}x^3 - 2x^{-3} + \pi^2$

$f'(x) = \frac{3}{4}x^2 + 6x^{-4}$

(b) [6 pts] $f(x) = \big(\arctan(x)\big)^3$

$f'(x) = 3(\arctan(x))^2 \cdot \frac{1}{1+x^2}$

(c) [8 pts] $f(x) = e^{\sqrt{x}\cos(x)}$

$f'(x) = e^{\sqrt{x}\cos(x)}\left(\sqrt{x} \cdot -\sin(x) + \cos(x) \cdot \frac{1}{2}x^{-1/2}\right)$

(d) [10 pts] $f(x) = x^{1/x}$

$\ln(f(x)) = \ln(x^{1/x})$

$\ln(f(x)) = \frac{1}{x}\ln(x)$

$\frac{d}{dx}$ $\frac{d}{dx}$

$\frac{1}{f(x)} f'(x) = \frac{1}{x}\left(\frac{1}{x}\right) + \ln(x)\left(\frac{-1}{x^2}\right)$

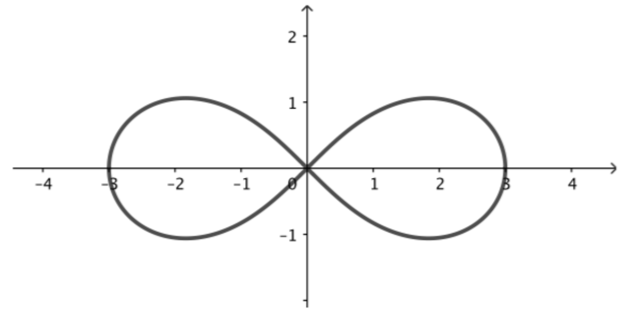
$f'(x) = f(x)\left(\frac{1}{x^2} - \frac{1}{x^2}\ln(x)\right)$

$f'(x) = x^{1/x}\left(\frac{1 - \ln(x)}{x^2}\right)$

4. Consider the curve defined by the equation

$$(x^2 + y^2)^2 = 9(x^2 - y^2).$$

The graph of the curve is provided to the right.



(a) [10 pts] Determine $\dfrac{dy}{dx}$.

$$2(x^2+y^2)\left(2x+2y\frac{dy}{dx}\right) = 9\left(2x-2y\frac{dy}{dx}\right)$$

$$2(x^2+y^2)(2x)+2(x^2+y^2)\left(2y\frac{dy}{dx}\right) = 18x-18y\frac{dy}{dx}$$

$$4x(x^2+y^2)+4y(x^2+y^2)\frac{dy}{dx} = 18x-18y\frac{dy}{dx}$$

$$4y(x^2+y^2)\frac{dy}{dx}+18y\frac{dy}{dx} = 18x-4x(x^2+y^2)$$

$$\frac{dy}{dx}\left(4y(x^2+y^2)+18y\right) = 18x-4x(x^2+y^2)$$

$$\frac{dy}{dx} = \frac{18x-4x(x^2+y^2)}{4y(x^2+y^2)+18y}$$

OR

$$4x^3+4xy^2+4x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx} = 18x-18y\frac{dy}{dx}$$

$$4x^2y\frac{dy}{dx}+4y^3\frac{dy}{dx}+18y\frac{dy}{dx} = 18x-4x^3-4xy^2$$

$$\frac{dy}{dx}\left(4x^2y+4y^3+18y\right) = 18x-4x^3-4xy^2$$

$$\frac{dy}{dx} = \frac{18x-4x^3-4xy^2}{4x^2y+4y^3+18y}$$

(b) [5 pts] Determine an equation of the line tangent to the curve at the point $(x, y) = (\sqrt{5}, 1)$.

$$\left.\frac{dy}{dx}\right|_{(\sqrt5,1)} = \frac{18\sqrt5-4\sqrt5(6)}{4(6)+18} = \frac{-6\sqrt5}{42} = \frac{-\sqrt5}{7}$$

$$\boxed{y-1 = \frac{-\sqrt5}{7}(x-\sqrt5)}$$

5. Determine the following indefinite integrals. **Please circle your final answer.**

(a) [6 pts] $\int (\cos(5x) + 1)\ dx = \frac{1}{5}\sin(5x) + x + C$

(b) [6 pts] $\int \left(\frac{1}{\sqrt{1-x^2}} + \frac{1}{e^x}\right)\ dx = \int\left(\frac{1}{\sqrt{1-x^2}} + e^{-x}\right)dx$

$= \arcsin(x) - e^{-x} + C$

OR $= -\arccos(x) - e^{-x} + C$

(c) [8 pts] $\int \frac{3}{x(\ln(x))^2}\ dx = \int 3(\ln(x))^{-2}\left(\frac{1}{x}\right)dx = \int 3u^{-2}du$

$u = \ln(x)$
$du = \frac{1}{x}dx$

$= 3(-1u^{-1}) + C$
$= -3(\ln(x))^{-1} + C$

6. Evaluate the following definite integrals. **Please circle your final answer.**

(a) [6 pts] $\int_0^1 \left(2x^2 + 3x + 5\right)\, dx = \left[\frac{2}{3}x^3 + \frac{3}{2}x^2 + 5x\right]_0^1$

$= \frac{2}{3} + \frac{3}{2} + 5$

$= \frac{4}{6} + \frac{9}{6} + \frac{30}{6}$

$= \frac{43}{6}$

(b) [8 pts] $\int_1^2 \left(\frac{x}{\sqrt[3]{x^2+1}}\right)\, dx = \frac{1}{2}\int_1^2 (x^2+1)^{-1/3}(2x)dx = \frac{1}{2}\int_2^5 u^{-1/3}du$

$u = x^2 + 1$
$du = 2x\, dx$
$u(1) = 2$
$u(2) = 5$

$= \frac{1}{2}\left[\frac{3}{2}u^{2/3}\right]_2^5$

$= \frac{3}{4}(5)^{2/3} - \frac{3}{4}(2)^{2/3}$

(c) [8 pts] $\int_0^{1/2} \left(\frac{e^{2x}}{1+e^{2x}}\right)\, dx = \frac{1}{2}\int_0^{1/2} \frac{1}{1+e^{2x}}(2e^{2x})dx = \frac{1}{2}\int_2^{1+e} \frac{1}{u}\, du$

$u = 1 + e^{2x}$
$du = 2e^{2x}dx$
$u(0) = 2$
$u(1/2) = 1 + e$

$= \frac{1}{2}\left[\ln|u|\right]_2^{1+e}$

$= \frac{1}{2}\ln(1+e) - \frac{1}{2}\ln(2)$

7. For this problem, consider the function $f(x) = |x-3|$.

(a) [5 pts] Use a Riemann Sum with 5 sub-intervals of equal width and left endpoints to estimate the definite integral $\int_0^4 |x-3|\,dx$.

$\Delta x = \frac{4-0}{5} = \frac{4}{5}$

$[0, 4/5]\ [4/5, 8/5]\ [8/5, 12/5]\ [12/5, 16/5]\ [16/5, 20/5]$ ← 4

$\int_0^4 |x-3|\,dx \approx f(0)\left(\frac{4}{5}\right) + f\left(\frac{4}{5}\right)\left(\frac{4}{5}\right) + f\left(\frac{8}{5}\right)\left(\frac{4}{5}\right) + f\left(\frac{12}{5}\right)\left(\frac{4}{5}\right) + f\left(\frac{16}{5}\right)\left(\frac{4}{5}\right)$

$= |0-3|\left(\frac{4}{5}\right) + \left|\frac{4}{5}-3\right|\left(\frac{4}{5}\right) + \left|\frac{8}{5}-3\right|\left(\frac{4}{5}\right) + \left|\frac{12}{5}-3\right|\left(\frac{4}{5}\right) + \left|\frac{16}{5}-3\right|\left(\frac{4}{5}\right)$ ← can stop here

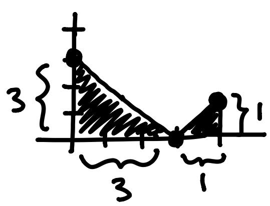
$= \frac{4}{5}\left(3 + \frac{11}{5} + \frac{7}{5} + \frac{3}{5} + \frac{1}{5}\right)$

$= \frac{4}{5}\left(\frac{15}{5} + \frac{22}{5}\right)$

$= \frac{4}{5}\left(\frac{37}{5}\right)$

$= \frac{148}{25}$

(b) [5 pts] Use areas to evaluate the definite integral $\int_0^4 |x-3|\,dx$.



$\int_0^4 |x-3|\,dx = \frac{1}{2}(3)(3) + \frac{1}{2}(1)(1)$ ← can stop here

$= \frac{10}{2}$

$= 5$

8. [10 pts] Use calculus to determine the absolute maximum and absolute minimum values of the function below on the interval $[-10, 1]$:

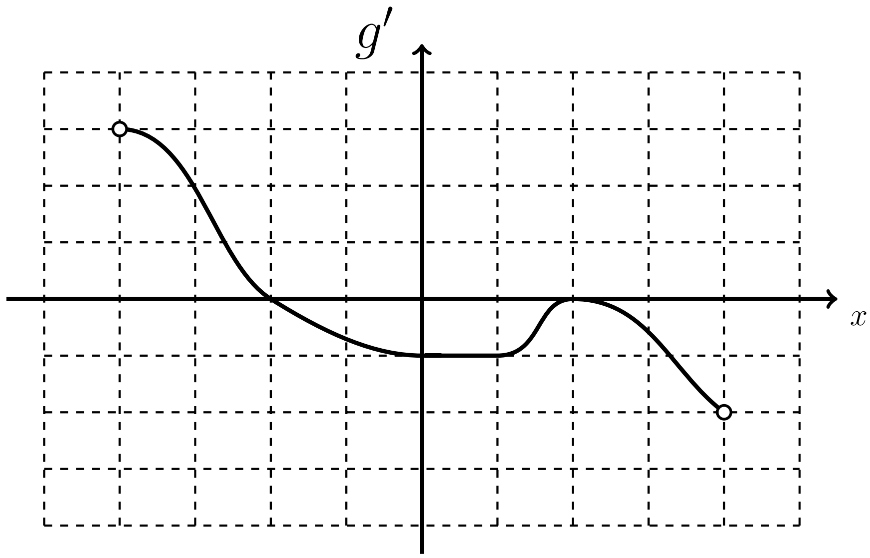
$$f(x) = \frac{x}{x^2 + 9}$$

$$f'(x) = \frac{(x^2+9)(1) - x(2x)}{(x^2+9)^2} = \frac{9-x^2}{(x^2+9)^2}$$

critical number: $9 - x^2 = 0$
$\cancel{x = \pm 3}$ (domain)
$x = -3$

x	$f(x) = \frac{x}{x^2+9}$
-10	$\frac{-10}{109}$
-3	$\frac{-3}{18} = -\frac{1}{6}$ ← abs min
1	$\frac{1}{10}$ ← abs max

9. The graph of $y = g'(x)$, the derivative of g, is given below. Consecutive grid lines are one unit apart.



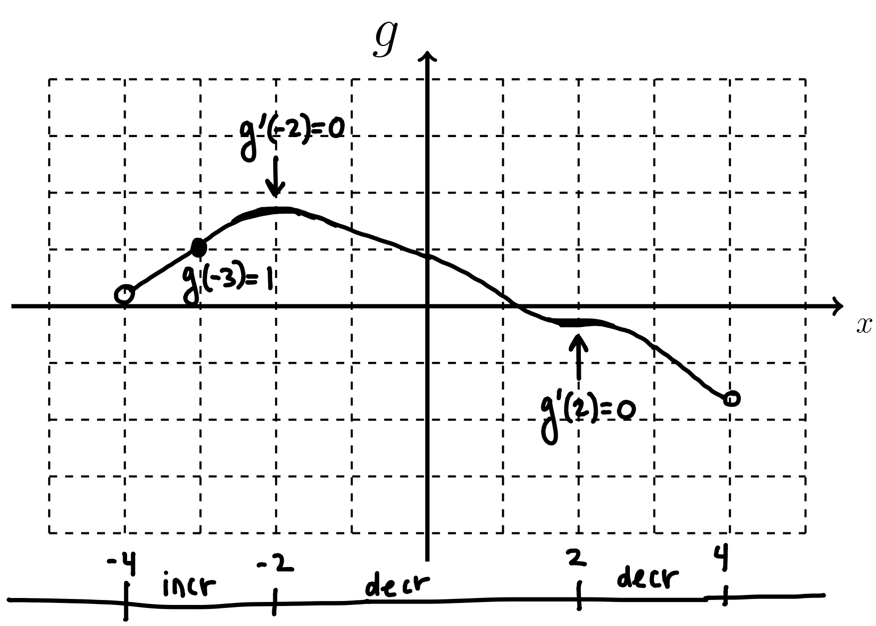
This is the graph of g'.

______ (a) [5 pts] State the intervals on which the function g is increasing. (Briefly explain your reasoning.)

$(-4,-2)$

The function g is increasing where $g'>0$.

______ (b) [5 pts] Assuming that $g(-3) = 1$, use the graph of g' to make a rough sketch of the graph of g on the axes below. Be sure to accurately show the increasing/decreasing behavior of g and any horizontal tangents of g on your graph.



10. [12 pts] On the axes below, sketch the graph of a function f having the following properties. (Consecutive grid lines are one unit apart.)

The domain of f is $(-\infty, \infty)$.

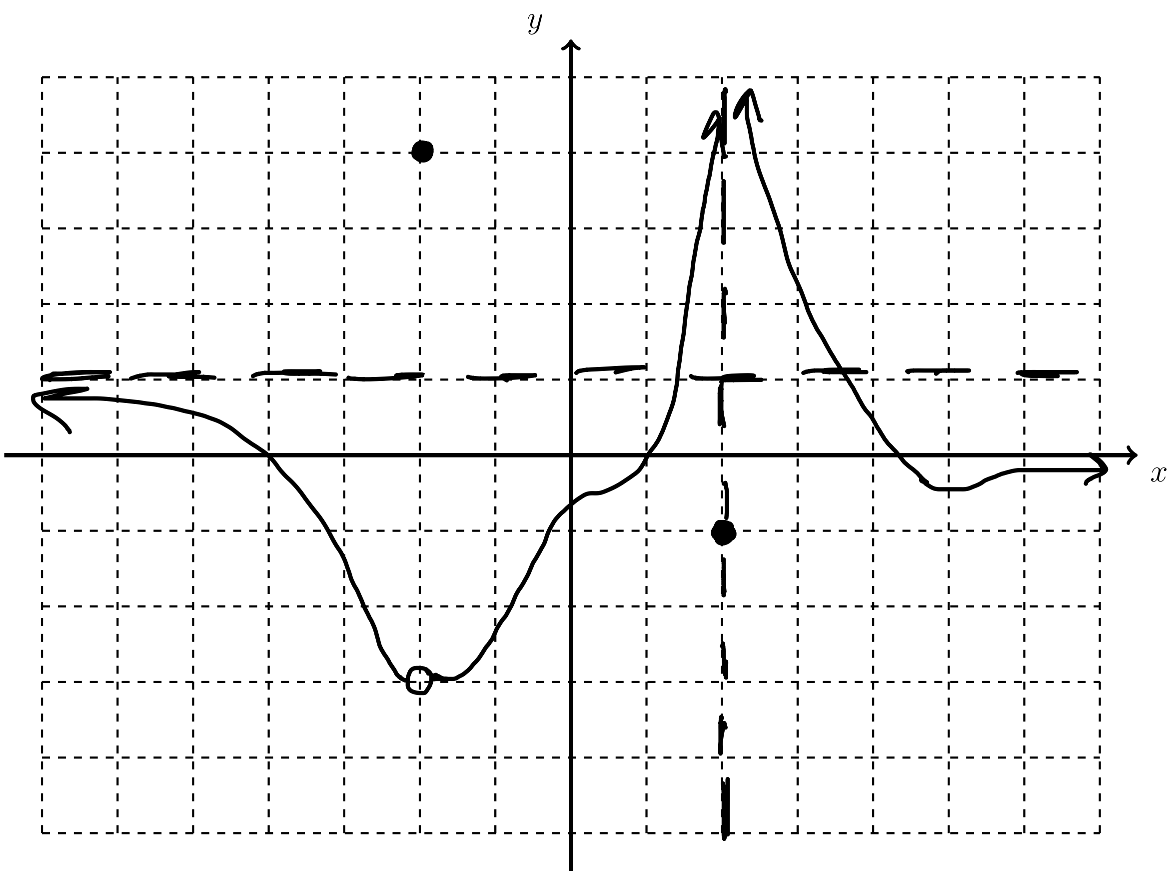
$f(-2) = 4$

$\lim_{x \to -2} f(x) = -3$

The line $y = 1$ is a horizontal asymptote.

$\lim_{x \to 2} f(x) = \infty$

$\lim_{x \to \infty} f(x) = 0$



11. Suppose f and g are continuous on $[0, 5]$; some values of $f(x)$, $f'(x)$, $g(x)$, and $g'(x)$ are given in the table below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-3	2	-1	-2
2	5	-1	-3	2
3	1	-3	4	-1

(a) [5 pts] Let $h(x) = f(x)g(x)$. Determine $h'(2)$.

$h'(2) = f(2)g'(2) + g(2)f'(2)$

$= 5(2) + (-3)(-1)$

$= 13$

(b) [5 pts] Determine the linearization $L(x)$ of $h(x) = f(x)g(x)$ at $x = 2$.

$h'(2) = 13$
$h(2) = f(2)g(2) = 5(-3) = -15$

$L(x) = h(a) + h'(a)(x-a)$

$L(x) = -15 + 13(x-2)$

(c) [5 pts] For the function $h(x) = f(x)g(x)$, use the linearization to approximate the value of $h(2.1)$.

$h(2.1) \approx L(2.1)$

$L(2.1) = -15 + 13(2.1 - 2) \leftarrow$ can stop here

$= -15 + 13(0.1)$

$= -15 + 1.3$

$= -13.7$

12. A particle moves along a straight line with position function $s(t) = \dfrac{15t^2}{2} - t^3$, for $t > 0$, where s is in feet and t is in seconds.

(a) [5 pts] Determine the velocity of the particle when the acceleration is zero.

$v(t) = 15t - 3t^2$
$a(t) = 15 - 6t$

$0 = 15 - 6t$
$6t = 15$
$t = \frac{15}{6} = \frac{5}{2}$

$\rightarrow v\left(\frac{5}{2}\right) = 15\left(\frac{5}{2}\right) - 3\left(\frac{5}{2}\right)^2$
$= \frac{75}{2} - \frac{75}{4}$
$= 75/4$ ft/second

(b) [5 pts] On the interval $(0, \infty)$, when is the particle moving in the positive direction? In the negative direction?

0 pos 5 neg

$0 = 15t - 3t^2$
$0 = 3t(5 - t)$
$t = 0,\ t = 5$ sec

$v(1) = 15 - 3 = 12$
$v(6) = 90 - 108 = -18$

$\boxed{\text{pos: } (0,5) \quad \text{neg: } (5,\infty)}$

(c) [5 pts] Determine all local (relative) extrema of the position function on the interval $(0, \infty)$. (You may use any relevant work from previous parts.)

t=5
pos neg

There's a local max (rel. max) at $t=5$. The local max is $s(5) = \frac{15}{2}(5)^2 - (5)^3$

(d) [5 pts] Determine $\dfrac{d}{dt}\left(\displaystyle\int_1^t s(u)\,du\right)$.

$= s(t)$ OR $\frac{15}{2}t^2 - t^3$

13. [15 pts] According to one model, the average global temperature x of the earth (in degrees Celsius) is related to the atmospheric concentration y of carbon dioxide (in parts per million, or ppm) by the equation

$$x^2 + 30x + 2125 = 140\sqrt{y}.$$

$x = 15$

When the global temperature is 15 degrees Celsius and the concentration of carbon dioxide is increasing at a rate of 5 ppm/year, determine the rate of change of the global temperature with respect to time.

$\hookrightarrow$ goal: $\frac{dx}{dt}$ $\qquad dy/dt = 5$

$$x^2 + 30x + 2125 = 140\sqrt{y}$$

$$2x\frac{dx}{dt} + 30\frac{dx}{dt} = 140\left(\tfrac{1}{2}y^{-1/2}\right)\frac{dy}{dt}$$

when $x=15$: need y

$$(15)^2 + 30(15) + 2125 = 140\sqrt{y}$$

$$\left(\frac{(15)^2 + 30(15) + 2125}{140}\right)^2 = y$$

$$\frac{dx}{dt}(2x+30) = 70\,\frac{1}{\sqrt{y}}\,\frac{dy}{dt}$$

$$\frac{dx}{dt} = \frac{70\frac{1}{\sqrt{y}}\frac{dy}{dt}}{2x+30} = \frac{70\frac{dy}{dt}}{2x+30}\left(\frac{1}{\sqrt{y}}\right)$$

when $x = 15$ and $\frac{dy}{dt} = 5$:

$$\frac{dx}{dt} = \frac{70(5)}{2(15)+30}\cdot\frac{140}{(15)^2+30(15)+2125} = \frac{70(5)}{60}\cdot\frac{140}{2800} = \frac{(7)(5)}{6}\cdot\frac{1}{20}$$

$$= \frac{7}{6\cdot 4} = \frac{7}{24}\ \text{deg C/year}$$

The global temp is increasing at a rate of $\frac{7}{24}$ deg. Celsius per year.

14. You plan to build a box with a square base and no lid. The material for the base of the box costs \$0.20 per square foot and the material for the sides costs \$0.05 per square foot. You can spend at most \$5.40 total on the materials for the box. You want to use these materials to construct the box of maximum possible volume which is within your budget.

_______ (a) [10 pts] Let x be the length, in feet, of each of the sides of the square base. Determine a formula for the function $V(x)$ representing the volume of the box having base dimension x feet. Your final answer $V(x)$ should include the variable x and may not include any other variables.

$V = x^2 y$

base cost: $(0.2)x^2$

total side cost: $4xy(0.05) = (0.2)xy$

constraint equation

$5.40 = 0.2(x^2 + xy)$

$54 = 2(x^2+xy)$

$27 = x^2 + xy$

$\frac{27-x^2}{x} = y$

$V = x^2\left(\frac{27-x^2}{x}\right)$

$V = x(27-x^2)$

$V = 27x - x^3$

either one

OR

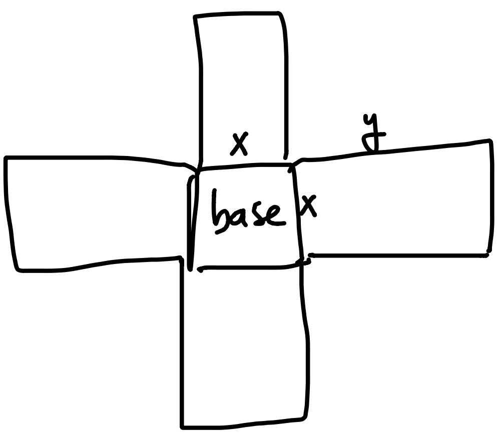
$5.40 = 0.2x^2 + 0.2xy$

$5.40 - 0.2x^2 = 0.2xy$

$\frac{5.40-0.2x^2}{0.2x} = y$

$V = x^2\left(\frac{5.4-0.2x^2}{0.2x}\right)$

$V = x\left(\frac{5.4-0.2x^2}{0.2}\right)$



_______ (b) [5 pts] Determine an appropriate domain for $V(x)$. Briefly explain your answer. (Note: There may be more than one correct answer; choose one domain you believe is a reasonable choice, and explain your thinking.)

constraint equation:

$27 = x^2 + xy$

$x \geq 0$ since it's a length

$x = 0$ violates constraint

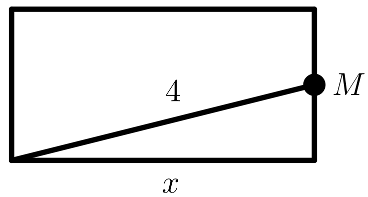
largest x: when $y = 0$

$27 = x^2$

$\sqrt{27} = x$

domain: $(0, \sqrt{27})$ or $(0, \sqrt{27}]$

15. A rectangle with base x units has a segment of length 4 cm joining one vertex to the midpoint M of an opposite side, as shown in the diagram below. The area $A(x)$ of the rectangle is given by the function $A(x) = 2x\sqrt{16 - x^2}$.



______ (a) [5 pts] Determine an appropriate domain for the function $A(x)$. Briefly explain your answer.

$[0,4]$ or $(0,4)$

allowing "flat rectangles"

not allowing "flat rectangles"

(b) [10 pts] Determine the maximum possible area of the rectangle. Use calculus to justify your answer.
______

$A(x) = 2x\sqrt{16-x^2}$

$A'(x) = 2x\left(\frac{1}{2}(16-x^2)^{-1/2}(-2x)\right) + \sqrt{16-x^2}\,(2)$

$A'(x) = \dfrac{-2x^2}{\sqrt{16-x^2}} + \dfrac{2(\sqrt{16-x^2})^2}{\sqrt{16-x^2}}$

$= \dfrac{-2x^2 + 2(16-x^2)}{\sqrt{16-x^2}}$

$= \dfrac{-2x^2 + 32 - 2x^2}{\sqrt{16-x^2}}$

$= \dfrac{32 - 4x^2}{\sqrt{16-x^2}}$

$= \dfrac{4(8-x^2)}{\sqrt{16-x^2}}$

A' dne: none in $(0,4)$

$A' = 0$: $x = \sqrt{8}$
($-\sqrt{8}$ not in domain)

using domain $[0,4]$:

extreme value theorem/closed interval method

x	$A(x)$
0	0
$\sqrt{8}$	$2\sqrt{8}\sqrt{8} = 16$ ← max area is 16 cm^2
4	0

using domain $(0,4)$:

1st derivative test

$(0, \sqrt{8})$ $A'(1) = \dfrac{4(7)}{\sqrt{15}}$

$(\sqrt{8}, 4)$ $A'(3) = \dfrac{4(-1)}{\sqrt{7}}$

Since A' changes from + to −, we know A changes from incr. to decr. and there is a local max at $x = \sqrt{8}$. Since there is only one critical # in $(0,4)$, the local max is absolute. The max area is $A(\sqrt{8}) = 16\text{ cm}^2$.

This page is extra space for work. **Do not detach this page.** If you want us to consider the work on this page, you should print your name, instructor and class meeting time below. For the problems where you want us to look at this work, please write "see last page" next to your work on the problem page so that we know to look here.

Name (print): ______________________ Instructor (print): ______________________

Class Meeting Time: ______________