The Pfaffian structure defining a Prym theta divisor

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Abstract: On the Prym variety of an étale double cover of curves, we construct locally a skew-symmetric matrix of regular functions whose Pfaffian is an equation for the theta divisor. This result accounts for several of the known features of the structure of a Prym theta divisor and its parametrization by an Abel map. The Pfaffian of the matrix of linear terms is the equation that Mumford introduced for the tangent cone to the Prym theta divisor at a point - as long as that polynomial is not identically zero. We illustrate the contribution of the higher order terms to the Pfaffian, and from Casalaina-Martin's multiplicity result we derive a Pfaffian equation for the tangent cone which holds whenever Mumford's equation is zero.

1. Introduction

Local Pfaffian equations for Prym theta divisors

It seems to be commonly assumed that in 1974 Mumford proved in [M2] that Kempf's determinantal equation for the tangent cone to a Jacobian theta divisor, induces a Pfaffian equation for the tangent cone to the theta divisor Ξ of the Prym variety P, associated to an étale double cover $\pi : \tilde{C} \to C$ of a curve of genus g. Some have assumed this as well for the restriction to P, of Kempf's matrix of locally regular functions whose determinant defines not the tangent cone but the theta divisor itself of the Jacobian of $\tilde{C}$ [K2, p. 160]. Nonetheless, to the best of our knowledge neither of these statements has been asserted in the literature up to now. In particular, Mumford did not address at all in [M2] the question of a (local) Pfaffian equation for the Prym theta divisor Ξ itself, and he makes clear there that the corresponding statement about tangent cones is sometimes false. For the literature on Pfaffian line bundles, we refer the reader to Plaza Martín's paper [PM] and the references there.

In the present paper we prove the existence of a local Pfaffian equation for the Prym theta divisor Ξ around every point, induced by restricting Kempf's determinantal equation for the Jacobian theta divisor of $\tilde{C}$. Then, using Casalaina - Martin's recent computation [CM] of the multiplicity of Ξ at every point L, we derive corresponding Pfaffian equations for the tangent cones to Ξ at all points, in terms of a local expansion of our equation for Ξ.

For a given curve C, Kempf defined a classifying map from a neighborhood of a point in $Pic^{g-1}(C)$ representing the line bundle L_0, to the space of $n \times n$ matrices where $n = h^0(C, L_0)$. Then the scheme W_{g-1} of effective line bundles in $Pic^{g-1}(C)$, is locally the pullback of the locus of singular matrices under this map. I.e. a local equation for W_{g-1} is given by the pullback of the universal determinant equation on square matrices.

Now consider the Prym variety P of the double cover $\tilde{C} \to C$, as a subvariety of $Pic^{\tilde{g}-1}(\tilde{C})$ and a point L on Ξ, where $2\Xi = \tilde{W}_{\tilde{g}-1} \cdot P$. Ideally the restriction to P of a local family of matrices, whose determinant equation defines $\tilde{W}_{\tilde{g}-1}$ locally near L, should give a local family of skew symmetric matrices whose Pfaffian equation defines Ξ locally near L. I.e. to get a local equation for Ξ around L, we want to first restrict the entries of the matrix of regular functions whose determinant defines $\tilde{W}_{\tilde{g}-1}$, and then prove

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this restricted matrix of regular functions is skew symmetric. This is indeed true, in an appropriate natural formulation, and is proved in this paper. Then the Pfaffian of this matrix provides a natural square root of the restricted equation, and hence a local equation for Ξ.

In terms of the local classifying map from line bundles to matrices, this says the classifying map defined around L in $Pic^{\tilde{g}-1}(\tilde{C})$, restricts on a neighborhood of L in the Prym variety P, to have values in the sublocus of skew symmetric matrices. Thus the scheme structure on Ξ is the pullback of the scheme of singular skew matrices defined by the pullback of the universal Pfaffian equation.

Pfaffian equations for tangent cones to a Prym theta divisor

By the result above, an equation for the tangent cone to Ξ at L, must be the non zero homogeneous form of lowest order, in the Pfaffian of the skew symmetric matrix M of restricted regular functions defining Ξ. Ideally this too is the Pfaffian of a matrix of suitable homogeneous forms from the entries of M. We recall however that it may differ from the Pfaffian of the matrix of linear forms from the entries of M. This means the derivative of the classifying map from the Prym variety to skew matrices can drop rank so much that it does not always pull back the tangent cone of singular skew matrices to the tangent cone of Ξ.

Nonetheless, we show there is still a Pfaffian equation for the tangent cone. In particular, both the Prym theta divisor Ξ, and all of its tangent cones, have an induced Pfaffian structure. I.e. the equation of the tangent cone to the Prym variety is the pullback of the universal Pfaffian equation by a matrix of certain homogeneous forms in the matrix defining Ξ. It turns out, instead of all linear forms, one takes some zero forms, some linear forms, and some quadratic forms.

We can summarize the situation for the tangent cone to Ξ at L as follows. Consider a matrix of regular functions whose determinant represents not a tangent cone, but the theta divisor itself of the Jacobian of $\tilde{C}$. Mumford's skew symmetric matrix contains only the restrictions of the linear terms from this matrix. If any of those linear terms come via pullback from the "downstairs curve" C, their restrictions are simultaneously symmetric and skew symmetric, hence zero. Thus if more than half of the linear terms come from the downstairs curve, as pointed out in [S] and generalized in [S-V5], then Mumford's restricted matrix has zero determinant.

Let A be the matrix of regular functions, whose determinant defines the Jacobian theta divisor $\tilde{W}_{\tilde{g}-1}$ of the upstairs curve $\tilde{C}$. Let B be the matrix of restrictions of these functions to the Prym variety, and let B_1 be the matrix of linear forms from the matrix B. If L is a point of Ξ, and M is a line bundle on C such that $\pi^*(M) \leq L$, and with $h^0(C, M)$ maximal, then one can choose the matrix A such that the matrix B_1 of restrictions of the linear forms, looks like this: $B_1 = \begin{bmatrix} 0 & \Lambda \\ -\Lambda^t & \kappa \end{bmatrix}$, where each block has linear forms as entries, the block κ is skew symmetric, and the block of zeroes is square of dimension $h^0(C, M)$.

Then the Pfaffian of the matrix B_1 defines the tangent cone to Ξ at L if and only if $h^0(C, M) \leq \frac{1}{2}h^0(\tilde{C}, L)$ [S-V5]. If $h^0(C, M) > \frac{1}{2}h^0(\tilde{C}, L)$ however, this Pfaffian is identically zero. In that case, let Q be the square matrix of dimension $h^0(C, M)$ consisting of the homogeneous quadratic terms of those regular functions in the restricted matrix

B belonging to the entries in the upper left block. Then Q is skew symmetric, and by Casalaina - Martin's multiplicity result, the lowest degree form in the Pfaffian of the skew symmetric matrix $\begin{bmatrix} Q & \Lambda \\ -\Lambda^t & \kappa \end{bmatrix}$ is an equation for the tangent cone of Ξ at L. This lowest degree term is itself then the Pfaffian of the matrix $\begin{bmatrix} Q & \Lambda \\ -\Lambda^t & 0 \end{bmatrix}$, so in all cases the tangent cone to the Prym theta divisor is defined by the Pfaffian of a skew symmetric $n \times n$ matrix of forms where $n = h^0(\tilde{C}, L)$.

We wish to point out that the Pfaffian structure presented here uses both the double cover and a choice of auxiliary divisor. We do not know in general to what extent it depends on this data, but for the intermediate Jacobian of a cubic threefold, we show the Pfaffian structure does vary with the double cover. It would be interesting to investigate in general the relation between the uniqueness of the Pfaffian structure and the Prym-Torelli problem.

The entries in Mumford's matrix B_1 of linear terms at L, are explicitly computed in [M2] following Kempf, from a basis of sections of $H^0(\tilde{C}, L)$. It would be desirable to give a geometric interpretation for the quadratic terms Q in our matrix, when Mumford's matrix has vanishing Pfaffian.

We give such an interpretation for the example of the intermediate Jacobian of a cubic threefold in terms of the conic bundle structure defining the Prym representation. The resulting Pfaffian equation for the tangent cone to Ξ (at the unique singular point): $x_0Q_0 - x_1Q_1 + x_2Q_2 = 0$, a homogeneous combination of linear and quadratic terms, is then an equation for the cubic threefold itself and displays the threefold as the union of a net of conics. Thus although the tangent cone to Ξ determines only the cubic threefold, the Pfaffian equation arising from a particular Prym representation determines also a choice of conic bundle structure for the threefold.

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2. History of the problem of Pfaffian equations for Prym tangent cones

Some cases where Mumford's Pfaffian tangent equation vanishes identically

Recall from [K1, Thm. 2, p. 185] Kempf's matrix of linear forms whose determinant defines the tangent cone at L to the theta divisor of the Jacobian of $\tilde{C}$, where $\pi : \tilde{C} \to C$ is an étale double cover. Mumford did show that if L is in Ξ, the restriction to the Prym tangent space at L, of that matrix of linear forms becomes skew symmetric, but that does not imply the corresponding Pfaffian defines the tangent cone. More precisely, his results there provide such a Pfaffian equation for the tangent cone to Ξ at L only under the extra assumption that $mult_L(\Xi) = \frac{1}{2}h^0(\tilde{C}, L)$, i.e. that the restricted Pfaffian does not vanish identically. Mumford explicitly says this [M2, p. 343, lines -7,-8], but he only examined this assumption when $h^0(\tilde{C}, L) = 2$.

Indeed Mumford was concerned only with identifying singular points of Prym theta divisors, i.e. points with $mult_L(\Xi) \geq 2$. Since in all cases we have $mult_L(\Xi) \geq \frac{1}{2}h^0(\tilde{C}, L)$, all points with $h^0(\tilde{C}, L) \geq 4$ are singular. He was thus interested only in finding those points L such that $h^0(\tilde{C}, L) = 2$, and yet $mult_L(\Xi) \geq 2$. Thus in fact his concern was to identify points L such that $h^0(\tilde{C}, L) = 2$ and his Pfaffian equation failed to hold, i.e. where it is identically zero. In that case, and that case only, he gave the necessary and sufficient condition that if L is a point of Ξ, then $mult_L(\Xi) = 1$, i.e. L is a smooth point of Ξ, if and only if $h^0(\tilde{C}, L) = 2$ and $h^0(C, M) \leq 1$ for every line bundle M on C such that $\pi^*(M) \leq L$, if and only if the Pfaffian determinant of the corresponding 2 by 2 skew symmetric matrix is non zero [M2, Prop., bottom of p. 343].

The principal goal of section III of [M2] is then to classify all possible examples of Prym theta divisors which have enough double points to resemble Jacobians of curves. By [M2, bottom p. 344, lemma p. 345] the main candidates are those Ξ with a lot of points L, such that C has an M with $\pi^*(M) \leq L$, and $h^0(C, M) \geq 2$. It can be shown that at most points of many such Ξ, Mumford's Pfaffian determinant is identically zero. E.g., for $g(C) \geq 7$, the only possibilities are that C is hyperelliptic, trigonal, or superelliptic [M2, Thm. (d), p. 344]. When C is hyperelliptic or trigonal, the Prym is either a hyperelliptic or tetragonal Jacobian, and the general such Jacobian arises this way [M2, Thm. (a), p.344; Re].

For a general such Jacobian $J(\Sigma)$, most singular points of the theta divisor are double points, and the tangent quadrics have base locus the canonical model of Σ [Gr; S-V1]. Since a tangent quadric at a double point of Ξ also contains the Prym canonical model of C if Mumford's Pfaffian equation holds [T, lemma 2.3, p. 963], that equation must fail at most singular points of such Ξ. Later Mumford mentions without proof the example of a Prym representation for the intermediate Jacobian of a cubic threefold, where $h^0(\tilde{C}, L) = 4$, but $mult_L(\Xi) = 3$, hence again his Pfaffian equation vanishes [M2, p. 348, lines 3-4].

In 1984 Shokurov extended the necessary condition for the validity of Mumford's Pfaffian tangent cone equation to the case $h^0(\tilde{C}, L) = 4$ [S, Lemma 5.7, p. 121], and deduced the Pfaffian vanishes at a large family of singular points of Ξ in the third major class of Mumford's examples, Pryms of "superelliptic" curves. Knowing most of these points are thus at least triple points, he deduced that such Pryms are not Jacobians of curves.

Thus the Pfaffian tangent cone equation in Mumford's original paper does not hold in general for the tangent cones of the Prym theta divisors which were the main subjects of his investigation: Jacobians of curves and intermediate Jacobians of cubic threefolds, and fails as well on a large locus of triple points for doubly covered superelliptic curves. In conversation with one of the present authors around 1980, Mumford implied that he knew an equation for the tangent cone in the case of the cubic threefold, but the published computations for that example [B; S-V2; F, ex. 4.3.2, p. 80, lines 7-15] have only established the set theoretic geometry of the tangent cone, and its multiplicity, without giving an explicit equation.

A criterion for Mumford's Pfaffian tangent equation to be valid

We have noted that essentially the only points where Mumford proved his Pfaffian equation valid are smooth points of Ξ. Next, his results imply that L on Ξ is a double point at which his Pfaffian equation holds if and only if $mult_L(\Xi) = 2$ and $h^0(\tilde{C}, L) = 4$, if and only if L is a "stable double point". These points play a key role in attempts to prove the Torelli theorem for Pryms, since as noted Tjurin showed the quadric tangent cones at such points always contain the Prym canonical model of C. The existence and characterization of stable double points is not immediate however. Tjurin gives a correct lower bound for the dimension of the locus of such points in general [T, p. 962, lines 8-10], but it seems neither the hypotheses nor the proof [T, Lemma 2.1, p. 961; T, Correction] are adequate. Not only a dimension count, but also non emptiness for the locus of points L on Ξ where $h^0(\tilde{C}, L) = 4$, must be established. After that it still remains to show at most such points that $mult_L(\Xi) = 2$, i.e. that the multiplicity is not higher.

In 1985 Welters proved that theta divisors on Prym varieties of generic doubly covered curves C do have a non empty locus of stable double points in codimension 5 when $g(C) \geq 17$ [W]. In 1987 Bertram extended this to non emptiness at least of the "stable singular locus" $= \{L \in \Xi | h^0(\tilde{C}, L) \geq 4\}$, for $g(C) \geq 7$, but did not settle the issue of the multiplicities of the points [Ber]. In 2002 we completed the precise characterization of stable double points by showing Shokurov's necessary condition for validity of Mumford's equation when $h^0(\tilde{C}, L) = 4$ is also sufficient [S-V4, Prop. 3.6, p. 246]. I.e. L on Ξ is a stable double point if and only if $h^0(C, L) = 4$ and $h^0(C, M) \leq 2$ for all M on C with $\pi^*(M) \leq L$, if and only if L is a double point where Mumford's Pfaffian gives an equation for the tangent quadric. Then the stable singular locus of Ξ is non empty of dimension $\geq g - 7$ whenever $g(C) \geq 7$, and stable double points are dense in that locus whenever $g(C) \geq 8$ and C is neither hyperelliptic, trigonal, nor superelliptic [S-V4, Th. 3.5, p. 246].

An analogous characterization of points of higher multiplicity where Mumford's Pfaffian equation holds, is announced in [S-V4] and proved in [S-V5]. The precise points at which Mumford's Pfaffian equation is non zero, and hence defines the Prym tangent cone, are those L on Ξ such that there is no line bundle M on C with $\pi^*(M) \leq L$ and $h^0(C, M) > \frac{1}{2}h^0(\tilde{C}, L)$. It then remained to give Pfaffian equations for the other tangent cones to Ξ, and to give such an equation for Ξ itself.

In the next section, we give a sketch of the well known determinantal structure for theta divisors of Jacobians, intended as an orientation to the topic. The sophisticated reader may wish to skip or peruse quickly this section. Afterwards we give a technical discussion of this material, including a self contained proof of the existence of a local Pfaffian equation for the Prym theta divisor, and a Pfaffian equation for the Prym tangent cones in all cases.

3. On the history of determinantal equations for theta divisors

Riemann's period matrix

Riemann's argument in his original paper on Abelian functions already shows that at least the locus C_d^r of divisors D of degree d and $dimL(D) > r$ on a given curve C, has a local determinantal structure, where $L(D) = \{$meromorphic functions f on C with $f = 0$ or $div(f) + D \geq 0\}$. I.e. his calculation of the linear conditions determining the dimension of $L(D)$, says that $dimL(D) - 1 = dimker[S(D)]$, where $S(D)$ is a (2g) by (g+d) "period matrix" for differentials of second kind, parametrized by the divisor D, [R1, p. 107, lines -3, -4; R2, p. 99, lines 3,4]. Thus C_d^r is the locus of divisors D such that $rank(S(D)) \leq$

$(d - r + g)$.

When $r = 1$, Riemann himself explicitly says this [R1, p. 108, lines 11-13; R2, p. 99, lines 14-16], and immediately concludes that a generic curve of genus g, is expected to have a non constant meromorphic function with at most d poles only if $d \geq (g/2) + 1$, [R1, p. 108, R2, p. 99], which he deduces from the inequality $(d-1) \geq (g+1-d)$ (= codimension of the rank (d-1+g) locus, in the space of (2g) by (g+d) matrices). The similar estimate $(d - r) \geq r(g + r - d)$ gives the "Brill - Noether" estimate for C^r_d to be non empty for all curves of genus g.

If we normalize Riemann's differentials of 2nd kind to have all "A - periods" equal to zero as Roch does, [Ro, p. 373, lines -13, -14], Riemann's period matrix becomes the g by d matrix $T(D)$ of just the "B - periods". This version is clearly stated for a modern reader in [G-H, p. 244], where the map at the bottom of p. 244 is the one represented by $T(D)$. We will also call this g by d period matrix "Riemann's matrix".

Roch's residue map

Roch's residue calculation of the matrix $T(D)$ equates it with a matrix of values of normalized differentials of first kind, evaluated at the points of the divisor D, [Ro, p.374, eq. (1), where $T(D) = -2\pi i[a^{(k)}_\mu]$; G-H, p. 245, top of page (up to sign)]. This allows Roch to restate Riemann's rank characterization of $dimL(D)$ in terms of values of holomorphic differentials, instead of periods of meromorphic differentials, with significant advantages both for computation, and for translation of the theorem into pure algebra.

On pp. 154-159 of [A-C-G-H], Arbarello et al. give Riemann's rank characterization of the loci C^r_d for all r using Roch's matrix of residues, which they call the "Brill Noether" matrix. Since the numerical entries in Roch's "evaluation" or "residue" matrix are proportional to those in Riemann's "period" matrix, this argument does not yet fully exploit the advantages of Roch's re - interpretation of the meaning of that matrix, as we wish to do next.

Since the statement $dimL(D) > r$, is invariant under linear equivalence of divisors, one may suspect a rank characterization should also exist for the corresponding loci of line bundles, but how is one to transfer the determinantal description of C^r_d, down to W^r_d = {line bundles L in $Pic^d(C)$ with $h^0(C, L) > r$}? Note the local "determinantal" structure for $C^0_d = C^{(d)}$ is given trivially by the zero equation, so for $d < g$, there is apparently no way to modify it to an equation for the proper subvariety $W^0_d = W_d$, in $Pic^d(C)$, in particular for the theta divisor W_{g-1}.

In both Riemann's and Roch's interpretations, the matrix $S(D)$ or $T(D)$ depends on the divisor D, while we need a matrix that depends only on $\mathcal{O}(D)$. It is interesting therefore that an examination of the duality between the two interpretations of T reveals how to modify it to eliminate the dependence on the divisor D as follows.

Sheaf theoretic versions of the Riemann and Roch maps

The sheaf version of Riemann's g by d period matrix is essentially the coboundary map $H^0(C, \mathcal{O}(D)|_D) \to H^1(C, \mathcal{O})$, induced from the sheaf sequence $0 \to \mathcal{O} \to \mathcal{O}(D) \to \mathcal{O}(D)|_D \to 0$, while Roch's evaluation map is the transpose of the restriction map $H^0(C, \mathcal{O}(K)) \to H^0(C, \mathcal{O}(K)|_D)$, induced from the sequence $0 \to \mathcal{O}(K - D) \to \mathcal{O}(K) \to \mathcal{O}(K)|_D \to 0$. This is fairly clear for Roch's map.

For Riemann's period map (in Roch's normalization), the source is the space $W(D)$

of meromorphic differentials with poles only at the points of D, of order at most 2 and with all residues zero, modulo holomorphic differentials. Now the sections $H^0(C, \mathcal{O}(D)|_D)$ of the skyscraper sheaf $\mathcal{O}(D)|_D$, is the space of possible principal parts for meromorphic functions with pole divisor supported in D, hence differentiation takes this space to a space of principal parts of differentials isomorphic to $W(D)$, by the converse of the residue theorem.

The target for Riemann's map is the orthogonal complement $(H_A)^\perp$ in $H^1(C, \mathbb{C})$ of the span H_A of the A - cycles in $H_1(C, \mathbb{C})$. Since the subspaces $(H_A)^\perp$ and $H^0(K)$ are complementary in $H^1(C, \mathbb{C})$, we may regard $(H_A)^\perp$ as naturally isomorphic to the quotient $H^1(C, \mathbb{C})/H^0(K) = H^1(C, \mathcal{O})$. Thus the period map on meromorphic differentials modulo holomorphic ones (i.e. normalized meromorphic differentials) goes into $H^1(C, \mathbb{C})/H^0(K) = H^1(C, \mathcal{O})$.

So the coboundary map $H^0(C, \mathcal{O}(D)|_D) \to H^1(C, \mathcal{O})$ may be thought of as the composition taking a principal part in $H^0(C, \mathcal{O}(D)|_D)$ by differentiation to a principal part for a meromorphic differential "of second kind", then to a unique such differential (modulo holomorphic ones), then to a cohomology class in $H^1(C, \mathbb{C})/H^0(K) = H^1(C, \mathcal{O})$.

Now the exact cohomology sequences coming from these two sheaf sequences are Serre dual to each other, i.e. the sequence (1) of maps:

$$0 \to H^0(C, \mathcal{O}) \to H^0(C, \mathcal{O}(D)) \to H^0(C, \mathcal{O}(D)|_D) \to H^1(C, \mathcal{O}) \to H^1(C, \mathcal{O}(D)) \to 0,$$

is dual to the following sequence (2):

$$0 \to H^0(\mathcal{O}(K - D)) \to H^0(\mathcal{O}(K)) \to H^0(\mathcal{O}(K)|_D) \to H^1(\mathcal{O}(K - D)) \to H^1(\mathcal{O}(K)) \to 0.$$

In particular, the two restriction spaces $H^0(\mathcal{O}(K)|_D)$ and $H^0(C, \mathcal{O}(D)|_D)$ are naturally dual, a fact that will be critical later, when this duality is used to state the condition of skew symmetry.

Geometrically, Riemann's map $H^0(C, \mathcal{O}(D)|_D) \to H^1(C, \mathcal{O})$ is the derivative at D of the Abel map $C^{(d)} \to Pic^d(C)$ from the symmetric product of C to the Picard variety, while Roch's dual map $H^0(\mathcal{O}(K)) \to H^0(\mathcal{O}(K)|_D)$, is the coderivative at D of the Abel map $C^{(d)} \to Alb(C)$ from the symmetric product to the Albanese variety. In this interpretation $H^0(C, \mathcal{O}(D)|_D)$ is the tangent space to $C^{(d)}$ at D, and $H^0(\mathcal{O}(K)|_D)$ is the cotangent space to $C^{(d)}$ at D. [Compare A-C-G-H, p. 160, and lemma 2.3, p. 171, where they refer to the coboundary version of Riemann's map also as the "Brill - Noether" homomorphism.]

The trick of adding a "base divisor"

Now look at the sequence (2) above as intended to compute, not the cohomology of D, but that of $L = K - D$, via the device of shifting by an auxiliary divisor D. Using this technique on an arbitrary line bundle L, we can shift it by some large divisor D, and use the exact sequence $0 \to H^0(L) \to H^0(L(D)) \to H^0(L(D)|D) \to H^1(L) \to 0$ arising from the sheaf sequence $0 \to L \to L(D) \to L(D)|_D \to 0$ to compute the cohomology of L, as kernel and cokernel of the evaluation map $H^0(L(D)) \to H^0(L(D)|_D)$. This is more efficient than sequence (2) itself, where one needs to know that $h^1(\mathcal{O}(K)) = 1$ in order to compute the cohomology of $\mathcal{O}(K-D)$, since now D may be chosen large enough so that $h^1(\mathcal{O}(L(D)))=0$.

A determinantal equation for the theta divisor $\mathbf{W_{g-1}}$

If we fix D so that $h^1(\mathcal{O}(L(D))) = 0$ for all L in Pic^d, this map $H^0(L(D)) \to H^0(L(D)|D)$ now depends only on the line bundle L, and we have given W_d^r as a rank locus, at least once we have a universal family of line bundles, i.e. a Poincaré bundle, on $C \times Pic^d(C)$. In particular, for $deg(L_0) = g-1$, and D a general divisor of degree $n = h^0(L_0) = h^1(L_0)$, the theta divisor $W_{g-1}^0 = W_{g-1}$ is defined locally for L near L_0, by the determinant of the $n \times n$ matrix of regular functions on $Pic^{g-1}(C)$, which represents the family of maps $H^0(L(D)) \to H^0(L(D)|_D)$. [Cf. K2, p. 31, p. 71; A-C-G-H pp. 176-177]. The point here is that sending a line bundle L to a matrix for the map $H^0(L(D)) \to H^0(L(D)|_D)$, defines a classifying map from a neighborhood of L_0 in $Pic^d(C)$, to matrix space. The scheme W_{g-1} is locally the pullback of the locus of singular matrices under this map, i.e. defined by the pullback of the universal determinant equation on $n \times n$ matrices, where $n = h^0(L_0)$. Moreover, since D is chosen so that $H^0(C, L_0) \to H^0(L_0(D))$ is an isomorphism, and $H^1(C, L_0(D)) = 0$, the point L_0 maps to the zero matrix representing $H^0(L_0(D)) \to H^0(L_0(D)|_D)$.

In fact, Riemann had already given a construction of a Poincaré line bundle for degree g divisor classes using his theta function. I.e. the pullback of $\mathcal{O}(W_{g-1})$ by the subtraction map $Pic^g(C) \times C \to Pic^{g-1}(C)$ is a universal line bundle of degree g, as Riemann's proof of "Jacobi inversion" shows, [K2, p. 154]. Indeed by translating to Pic^{g-1} Riemann proved his theta function vanishes exactly on W_{g-1}. Kempf suggests that by taking matrices of quotients of theta functions, Riemann could also have given a determinantal characterization of W_{g-1} [K2, p. 166].

Determinantal equations for the tangent cones to $\mathbf{W_{g-1}}$

Now that we have a determinantal equation for W_{g-1}, how do we compute an equation for its tangent cone at a given point L? By definition we want the lowest degree non zero homogeneous form of the determinantal equation for W_{g-1}, which a priori may not itself be the determinant of the matrix of lowest degree terms. Of course a natural candidate is the determinant of the linear forms of the entries in the $h^0(L)$ by $h^0(L)$ matrix whose determinant defines W_{g-1}, but to prove that, one must show this determinant is not zero. It would suffice for example to know the multiplicity of the point is equal to the degree of this determinant of linear forms, i.e. to know Riemann's singularity theorem. This is equivalent to showing that the tangent cone of W_{g-1} at L is defined by the universal determinant for $n \times n$ matrices, where $n = h^0(L)$, i.e. the tangent cone at the zero matrix, after pullback by the derivative at L of the classifying map above.

A more direct approach, without using the determinantal equation for W_{g-1}, is simply to write down the appropriate determinant of linear forms, prove it is non zero, reduced and irreducible, and vanishes on the tangent scheme to W_{g-1} at L. Kempf does this in [K1], when he shows that if $\{w_i\}$ is a basis for $H^0(C, \mathcal{O}(L))$ and $\{z_j\}$ is a basis for $H^0(C, \mathcal{O}(K-L))$, then the determinant of the matrix $[w_i \cdot z_j]$ is non zero and vanishes on the image of the normal cone in $C^{(g-1)}$ to the linear series $|L|$, under the derivative of the Abel map $C^{(g-1)} \to W_{g-1} \subset Pic^{g-1}(C)$. This idea apparently goes back to Andreotti and Mayer [A-M], and we learned it from Mayer about 1968. Kempf also shows his tangent equation is the pullback of the universal determinant by the derivative of the classifying map; see [K2, §18]. Even stronger [cf. K1, Theorem, p. 183], the local classifying map from $Pic^{g-1}(C)$ to the space of square matrices is transverse to the standard map resolving the locus of singular matrices, since the divisor variety C^{g-1} is smooth in its pulled back scheme structure.

The case of Prym varieties

Now how does all this descend to the Prym case? The Prym theta divisor Ξ in P in $Pic^{\tilde{g}-1}(\tilde{C})$, is defined so that $\tilde{W}_{\tilde{g}-1}$ meets P in the divisor $2\cdot\Xi$, i.e. the local determinantal equation for $\tilde{W}_{\tilde{g}-1}$ restricts to the square of a local equation for Ξ. Hence the lowest order non zero homogeneous form of this restriction is the square of an equation for the tangent cone to Ξ at L.

Since the Prym variety is the skew symmetric part of the Jacobian $Pic^{\tilde{g}-1}(\tilde{C})$ under the involution induced by the double cover $\pi : \tilde{C} \to C$, and since the determinant of a skew symmetric matrix is the square of its Pfaffian, the most natural explanation for this situation would be that both the Prym theta divisor and its tangent cones are given by Pfaffians. We prove these results along the lines indicated for Jacobians. One must make explicit in what sense the map $H^0(L(D)) \to H^0(L(D)|_D)$ becomes skew symmetric for L on the Prym theta divisor. With some care taken in the choice of the auxiliary divisor D, a natural residue pairing is nondegenerate on a neighborhood of a given point $L_0 \in \Xi$ and everything works out as might be expected.

4. Brief technical review of the determinantal structure defining a Jacobian theta divisor

Let C be a connected, smooth, projective curve of genus $g \geq 1$. Set $J = Pic^{g-1}(C)$ and $\Theta = \{L \in J \mid h^0(L) > 0\}$ (also known as $W_{g-1}(C)$). We will use these models for the Jacobian variety and its theta divisor, and then the Abel map $C^{(g-1)} \to \Theta \subset J$ parametrizing the Jacobian theta divisor is defined by $D \mapsto \mathcal{O}_C(D)$ for $D \in C^{(g-1)}$.

Let $\mathcal{P}$ be a Poincaré line bundle on $C \times J$, that is, any line bundle on the product such that the mapping $J \to Pic(C)$, defined by $L \mapsto \mathcal{P}|_{C\times\{L\}}$ (on $C \times \{L\} \cong C$), is simply the inclusion of $J = Pic^{g-1}(C)$ in $Pic(C)$. Now let D be an effective divisor on C and consider the corresponding divisor $D \times J \subset C \times J$ on the product. Then one gets the exact sequence

$$0 \to \mathcal{P} \to \mathcal{P}(D \times J) \to \mathcal{P}(D \times J)|_{D\times J} \to 0$$

on $C \times J$. Let $q : C \times J \to J$ be the projection, and push these sheaves forward by q. One obtains on J an exact sequence

$$0 \to q_*(\mathcal{P}(D \times J)) \to q_*(\mathcal{P}(D \times J)|_{D\times J}) \to R^1q_*(\mathcal{P}) \to R^1q_*(\mathcal{P}(D \times J)) \to 0$$

since $q_*(\mathcal{P}) = 0$ ($H^0(L) = 0$ for $L \in J - \Theta$) and $R^1q_*(\mathcal{P}(D \times J)|_{D\times J}) = 0$ ($q : D \times J \to J$ is a finite morphism).

If the divisor D has sufficiently large degree (specifically, $deg(D) \geq g$, so that $H^1(L(D))$ $= 0$ for all $L \in J$), then $q_*(\mathcal{P}(D \times J))$ is a vector bundle on J and $R^1q_*(\mathcal{P}(D \times J)) = 0$.

In this situation, if we set $\mathcal{E} = q_*(\mathcal{P}(D \times J))$, $\mathcal{F} = q_*(\mathcal{P}(D \times J)|_{D\times J})$, and $\mathcal{R} = R^1q_*(\mathcal{P})$, then we have the exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{R} \to 0$$

of coherent sheaves on J. The key point is that for every $L \in J$, the cohomology vector spaces $H^0(C, L)$ and $H^1(C, L)$ now occur as the kernel and cokernel of the linear map $H^0(C, L(D)) \to H^0(C, L(D)|_D)$ induced pointwise on fibers at L by the homomorphism from $\mathcal{E}$ to $\mathcal{F}$. The 2-term complex $\mathcal{E} \to \mathcal{F}$ is a map of equal rank vector bundles whose associated determinant line bundle and section on J define the Jacobian theta divisor $\Theta \subset J = Pic^{g-1}(C)$.

Recall that for a homomorphism $h : \mathcal{E} \to \mathcal{F}$ of rank r vector bundles, the determinant line bundle is $\Lambda^r(\mathcal{E})^* \otimes \Lambda^r(\mathcal{F})$ and its (determinant) section corresponds to the induced homomorphism $\Lambda^r(h) : \Lambda^r(\mathcal{E}) \to \Lambda^r(\mathcal{F})$ on the top exterior powers. In other words, the resulting section of the line bundle $\Lambda^r(\mathcal{E})^* \otimes \Lambda^r(\mathcal{F})$ is an abstract, basis free, determinant for h. Also note that the structure of a vector bundle map contains more information than just the determinant line bundle and its section. For instance, the map $\mathcal{O}_J \to \mathcal{O}_J(\Theta)$ of equal rank vector bundles defines the same determinant line bundle and section, but even locally is not the "correct" determinantal structure for defining the Jacobian theta divisor since at singular points of Θ, the cokernel is different from $\mathcal{R} = R^1q_*(\mathcal{P})$. For another example, the dual vector bundle homomorphism $\mathcal{F}^* \to \mathcal{E}^*$ has the same determinant line bundle and section, but may have a different cokernel.

Now we state a detailed local description of the well-known determinantal structure for a Jacobian theta divisor. All of the properties given here are treated in great detail in [K2] and can also be found in [A-C-G-H]. To obtain this version around a given point $L \in \Theta$, one follows the same route as indicated, but with the following refinement. If $h^1(L) = n$, then one can take an effective divisor D of degree n on C such that $H^1(L(D)) = 0$ and obtain the sequence $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{R} \to 0$ on a neighborhood of L in J. Then trivializing the rank n vector bundles $\mathcal{E}$ and $\mathcal{F}$ on a neighborhood of L will represent the map $\mathcal{E} \to \mathcal{F}$ by an $n \times n$ matrix (of regular functions).

Theorem (Mumford and Kempf): Fix $L \in \Theta \subset J$ and let $n = h^0(L)$. Then there exists an $n \times n$ matrix $\Phi = (\varphi_{ij})$ of regular functions on a neighborhood U of L in J such that $det(\Phi)$ is a local equation for Θ in J around L. Moreover,
(1) the matrix Φ_1 of linear terms at L has an intrinsic interpretation as the dual of the "Petri map": $H^0(L) \otimes H^0(K \otimes L^*) \to H^0(K)$ defined by cup product. Of course, the matrix Φ_1 has entries $\lambda_{ij} = d\varphi_{ij}|_L$ and represents the derivative at L of the map Φ from U to the space of $n \times n$ matrices. In fact, Φ_1 is a "Riemann-Kempf matrix" for the tangent cone $C_L(\Theta)$, in the sense that there exist bases $\{\sigma_1, ..., \sigma_n\}, \{\tau_1, ..., \tau_n\}$ of $H^0(L), H^0(K \otimes L^*)$ resp. such that $\lambda_{ij} = \sigma_i \cdot \tau_j$ in $H^0(K) \cong T_L^*(J)$ and $\det(\Phi_1)$ is an equation for $C_L(\Theta)$.
(2) the Abel map $C^{(g-1)} \to \Theta \subset J$ is induced in U from the family of projectivized kernels of the singular matrices by pullback via Φ.

Examples:
(i) double points of Θ. If $h^0(L) = 2$, then

$$\Phi = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } det(\Phi) = ad - bc$$

where a, b, c, d are regular functions vanishing at L. Thus, Θ is defined locally by $ad - bc = 0$. If the double point has rank 4, then the differentials of these functions are linearly independent linear forms, say x, y, z, w, and the tangent cone $C_L(\Theta)$ has equation $xw - yz = 0$.

Instead if the double point has rank 3, with tangent cone $x^2 - yz = 0$, then knowing about the higher order terms in Φ is essential for determining the local structure of Θ. Indeed, if we include $a = x, b = y, c = z$ in a local (formal) coordinate system, then $d = x + \Delta$ (where Δ consists of the terms of order at least 2) and the local equation for Θ is $x^2 - yz + x\Delta = 0$. Then completing the square by writing $x = \tilde{x} + \frac{\Delta}{2}$, the 2×2 matrix becomes $\Phi = \begin{bmatrix} \tilde{x} & y \\ z & \tilde{x} \end{bmatrix} + \begin{bmatrix} Q & 0 \\ 0 & -Q \end{bmatrix} + \ldots$, and the determinant takes the form $\tilde{x}^2 - yz + Q^2 + \ldots$, where Q is a quadratic form (and the dots indicate higher order terms). It seems surprizing to get such a restricted form in this case for the local equation defining $\Theta \subset J$.

(ii) triple points of Θ. If $h^0(L) = 3$ and the Petri map for L is injective, then the singular scheme $Sing(\Theta)$ is reduced at L. Here, the scheme structure on the singular locus of Θ is defined by a local equation ϑ for the hypersurface $\Theta \subset J$ and the partials of ϑ. Indeed, the local map from Θ to the space of 3×3 matrices is submersive, so it suffices to check the structure at 0 of the singular scheme of the hypersurface of 3×3 matrices with determinant zero. The equations for this singular scheme are simply the collection of all 2×2 minors, so this scheme is the cone over the standard Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ and is quite singular at the origin, but is reduced (and Cohen-Macaulay). (In this connection, we would like to thank O. Debarre for inquiring years ago about something misleading we wrote about triple points of Jacobian theta divisors, and we thank P. Aluffi for some discussion of triple points of hypersurfaces.)

5. The local Pfaffian structure defining a Prym theta divisor

Now we head towards our formulation of the local existence of a "Pfaffian structure" on the Prym variety of an étale double cover. We work over an algebraically closed field k of characteristic $\neq 2$. Let C be a connected, smooth, projective curve of genus $g \geq 1$, and let $\pi : \tilde{C} \to C$ be a connected étale double cover. Then $\tilde{C}$ has genus $\tilde{g} = 2g - 1$, and we let $\tilde{J} = Pic^{\tilde{g}-1}(\tilde{C})$ denote the Picard variety of line bundles of degree $\tilde{g} - 1$. Our models for the Prym variety and its theta divisor are $P = \{L \in \tilde{J} \mid Nm(L) \cong K_C \text{ and } h^0(L) \text{ is even}\}$ and $\Xi = \{L \in P \mid H^0(\tilde{C}, L) \neq 0\}$.

Theorem: Let (P, Ξ) be the Prym variety of an étale double cover $\pi : \tilde{C} \to C$. Let L be a point of the Prym theta divisor $\Xi \subset P$, and set $n = h^0(\tilde{C}, L)$. Then there exists an open neighborhood U of L in P and a skew-symmetric $n \times n$ matrix M of regular functions on U vanishing at L such that the Pfaffian of M is an equation in U for the divisor Ξ. Moreover,

(1) the skew-symmetric matrix M_1 of linear terms at L has an intrinsic interpretation as the dual of the Prym-Petri map, and
(2) the Abel map $X \to \Xi \subset P$ is induced in U from the family of projectivized kernels of the singular skew-symmetric matrices by pullback via M.

Before proceeding with the proof, let us explain the precise meaning of (1) and (2). The derivative at L of the inducing map $M = (f_{ij}) : U \to Alt(n) = \{$skew-symmetric $n \times n$ matrices$\}$ is given by the matrix M_1 of linear terms $\omega_{ij} = df_{ij}|_L$. Now we make the identification $T_L(U) \cong T_0(P) = H^0(C, K_C(\eta))^*$, where η is the $\frac{1}{2}$-period defining the given étale double cover of C. Since $Alt(n)$ can be regarded as the alternating bilinear forms on $k^n \cong H^0(L)$, we equate the tangent space $T_{M_L}(Alt(n))$ with $\Lambda^2(H^0(L)^*)$, the alternating bilinear forms on $H^0(L)$. Then the derivative $dM|_L$ is a linear map $H^0(C, K_C(\eta))^* \to \Lambda^2(H^0(L)^*)$ and the dual linear map $\Lambda^2(H^0(L)) \to H^0(C, K_C(\eta))$ is the "Prym-Petri map" as defined in [W, p. 673; cf. M2, p. 343].

Let $X = \{D \in \tilde{C}^{(2g-2)} \mid Nm(D) \in |K_C|$ and $h^0(D)$ is even$\}$, and let $\varphi : X \to \Xi$ be the restriction to $X \subset \tilde{C}^{(2g-2)}$ of the standard Abel map $\tilde{C}^{(2g-2)} \to P$ defined by $D \mapsto \mathcal{O}_{\tilde{C}}(D)$. Then there is an isomorphism $\varphi^{-1}(U) \cong \{(u, [v]) \in U \times \mathbb{P}^{n-1} \mid M_u v = 0\}$, commuting with the projections to U; in particular, for $u = \varphi(D) = \mathcal{O}_{\tilde{C}}(D)$, the fibre of the Abel map over u is $\varphi^{-1}(u) = |D| \cong \mathbb{P}(ker M_u)$.

Proof: Fix the notation L_0 for the given point of Ξ so that we can use "L" for a variable point of P. We will give the construction in stages and indicate the successive assumptions, keeping the neighborhoods of L_0 in P as large as possible.

(i) We will work with the restriction to the Prym variety of the vector bundle homomorphism $\mathcal{E} \to \mathcal{F}$ constructed for the Jacobian case on $\tilde{J} = Pic^{2g-2}(\tilde{C})$. However, this time we need to arrange a particular normalization of the Poincaré bundle in order for the pointwise isomorphisms $\pi^*(Nm(L)) \cong \tilde{K} = K_{\tilde{C}}$ to work globally on P. (Note the ϕ Mumford uses in [M2], the footnote in [D-P, p. 688], and [F-P, p. 93].) Though such a normalization is part of our construction, once we find the normalization we will present most of the calculations pointwise for simplicity. At any rate, the condition that $Nm(L) \cong K_C$ on C (which is stronger than $\pi^*(Nm(L)) \cong \tilde{K}$ on $\tilde{C}$) is essential in our proof of skew-symmetry of the vector bundle homomorphism $\mu : \mathcal{A} \to \mathcal{A}^*$ constructed below.

For any Poincaré line bundle $\mathcal{P}$ on $\tilde{C} \times \tilde{J}$, consider the restriction $\mathcal{Q} = \mathcal{P}|_{\tilde{C} \times P}$ to $\tilde{C} \times P$, and the map $(\pi, id) : \tilde{C} \times P \to C \times P$. What we need to specify is an appropriate choice of $\mathcal{P}$ admitting an isomorphism $\phi : (\pi, id)^*(Nm_{(\pi,id)}(\mathcal{Q})) \xrightarrow{\sim} p^*(\tilde{K})$, where $p : \tilde{C} \times P \to \tilde{C}$ is the projection. We will indicate the geometry leading in fact to an isomorphism $Nm_{(\pi,id)}(\mathcal{Q}) \xrightarrow{\sim} p_C^*(K_C)$, where p_C is the projection $C \times P \to C$.

Any line bundle on the product $C \times P$ defines maps

$$\left.\begin{array}{l} C \to Pic(P) \\ P \to Pic(C) \end{array}\right]$$

and is determined by this pair of maps. To analyze what is involved, note first that the two line bundles $Nm_{(\pi,id)}(\mathcal{Q})$ and $p_C^*(K_C)$ on $C \times P$ both define the constant map $P \to Pic(C)$ with value K_C, by construction. Thus we want to adjust $\mathcal{P}$ by the pullback of a line bundle from $\tilde{J}$ so that the morphism $C \to Pic(P)$ defined by $Nm_{(\pi,id)}(\mathcal{Q})$ coincides with the 0-map defined by $p_C^*(K_C)$.

It is well known that the identity component of $Pic(\tilde{J})$ is canonically isomorphic to $\tilde{J}_0 = Pic^0(\tilde{C})$ and $\mathcal{P}$ may be chosen so that the induced map $a : \tilde{C} \to Pic(\tilde{J})$ is given by $a(x) = \mathcal{O}_{\tilde{C}}(x - x_0)$ for some base point $x_0 \in \tilde{C}$. It follows that the map $b : \tilde{C} \to Pic(P)$ induced by $\mathcal{Q}$ is given by $b(x) = \mathcal{O}_{\tilde{C}}(x - x') \otimes b_1$, where $b_1 = \mathcal{O}_{\tilde{C}}(x_0 - x_0') \in Pic^0(\tilde{C})$. (Here $x \mapsto x' = \iota(x)$ is the involution on $\tilde{C}$ corresponding to the double covering map π.) Therefore the map $C \to Pic(P)$ defined by $Nm_{(\pi,id)}(\mathcal{Q})$ is the constant map $y \mapsto b(x) + b(x') = 2b_1$ for $y \in C$ and $\pi^{-1}(y) = \{x, x'\}$. Then since $2b_1$ (but not b_1 itself!) is in the identity component P_0 of the kernel of $Nm : \tilde{J}_0 \to J_0 = Pic^0(C)$ and $id - \iota^* : \tilde{J}_0 \to P_0$ is surjective, there exists $a_1 \in \tilde{J}_0$ such that $2(a_1 - a_1') = 2b_1$ (where $a_1' = \iota^*(a_1)$). Thus by subtracting a_1 we can adjust $\mathcal{P}$ as desired.

Having normalized the Poincaré line bundle $\mathcal{P}$ on $\tilde{C} \times \tilde{J}$, let D be an effective divisor on $\tilde{C}$ such that $H^1(\tilde{C}, L_0(D)) = 0$ and construct as in the previous section an exact sequence

$$0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{R} \to 0$$

on the open neighborhood $\{L \in \tilde{J} \mid H^1(\tilde{C}, L(D)) = 0\}$ of L_0 in $\tilde{J}$, where $\mathcal{E} = q_*(\mathcal{P}(D \times \tilde{J}))$ and $\mathcal{F} = q_*(\mathcal{P}(D \times \tilde{J})|_{D \times \tilde{J}})$ are vector bundles of rank $= deg(D)$, and $\mathcal{R} = R^1 q_*(\mathcal{P})$ is a coherent sheaf. Now let

$$0 \to \mathcal{A} \xrightarrow{h} \mathcal{B} \to \mathcal{S} \to 0$$

be the induced exact sequence of sheaves on the open neighborhood $U_1 = \{L \in P \mid H^1(L(D)) = 0\}$ of L_0 in P, where $\mathcal{A}$ and $\mathcal{B}$ are respectively the vector bundles $\mathcal{E}|_{U_1}$ and $\mathcal{F}|_{U_1}$ and $\mathcal{S}$ is the coherent sheaf $\mathcal{R}|_{U_1}$. Note that the map h is injective as a sheaf homomorphism since $\mathcal{A}$ is a vector bundle and h is injective on a generic fibre. Indeed, h induces an isomorphism of vector bundles on the complement of Ξ in P since the sheaf sequence $\mathcal{A} \to \mathcal{B} \to \mathcal{S} \to 0$ is exact, $\mathcal{S}$ is 0 outside Ξ, and $\mathcal{A}$ and $\mathcal{B}$ have the same rank.

(ii) With the above notation we have a homomorphism $h : \mathcal{A} \to \mathcal{B}$ of vector bundles on $U_1 = \{L \in P \mid H^1(L(D)) = 0\}$. (If $deg(D) \geq \tilde{g}$, then $U_1 = P$.) At $L \in U_1$ the two vector bundles have fibres $\mathcal{A}|_L = H^0(L(D))$ and $\mathcal{B}|_L = H^0(L(D)|_D)$, and the linear map h_L on the fibres is simply the natural restriction map $H^0(L(D)) \to H^0(L(D)|_D)$ from the global sections of $L(D)$ to their values on $D \subset \tilde{C}$.

Now assume that the effective divisor D on $\tilde{C}$ consists of distinct points, and that D and its conjugate D' (under the involution on $\tilde{C}$) are disjoint. We will define a vector bundle homomorphism $\beta : \mathcal{B} \to \mathcal{A}^*$ so that the composition $\mu = \beta \circ h : \mathcal{A} \to \mathcal{A}^*$ is skew-symmetric. Of course the dual vector bundle $\mathcal{A}^*$ has fibres $\mathcal{A}^*|_L = H^0(L(D))^*$ and for simplicity we give the prescription for β pointwise on U_1.

Thus we want to define a linear map:

$$H^0(L(D)|_D) \xrightarrow{\beta_L} H^0(L(D))^*$$

and it suffices to specify the corresponding bilinear pairing:

$$H^0(L(D)|_D) \times H^0(L(D)) \xrightarrow{B_L} k$$

We let ι denote the involution of the double cover $\tilde{C}$ and use D' (resp. L') to denote the conjugate of a divisor D (resp. line bundle L) with respect to the action of ι.

Definition of B_L: Take $s \in H^0(L(D)|_D)$ and $\tau \in H^0(L(D))$, and proceed as follows. First map $(s,\tau) \mapsto (s,\iota^*(\tau)) \mapsto s\otimes\iota^*(\tau)|_D \in H^0(L(D)|_D\otimes L'(D')|_D) \cong H^0((L\otimes L')(D+D')|_D)$. Next use the isomorphism $\phi : L\otimes L' \xrightarrow{\sim} \tilde{K}$ to obtain $\phi(s\otimes\iota^*(\tau)|_D) \in H^0(\tilde{K}(D+D')|_D)$, and then finally apply $r_D : H^0(\tilde{K}(D+D')|_D) \to k$, ending up with the scalar $\mathrm{r}_D(\phi(s\otimes\iota^*(\tau)|_D))$. Here r_D denotes the composite map $H^0(\tilde{K}(D+D')|_D) \cong H^0(\mathcal{O}_D) \xrightarrow{sum} k$, where we have used $\tilde{K}(D)|_D \cong (\tilde{K}|_D)\otimes\mathcal{O}_D(D) \cong O_D$ (from the natural duality signaled in section 3 after the long exact sequences (1) and (2)) and $\mathcal{O}(D')|_D \cong \mathcal{O}_D$ (since D and D' are disjoint). Note that for $\omega \in \mathrm{H}^0(\tilde{K}(D+D'))$, $r_D(\omega|_D) = Res_D(\omega)$, the sum of the residues of the rational differential ω at points of D.

Let's look back at the various steps in the procedure defining $B_L(s,\tau)$. We start with $(s,\tau) \in H^0(L(D)|_D) \times H^0(L(D))$ and successively apply the mappings $H^0(L(D)|_D) \times H^0(L(D)) \cong H^0(L(D)|_D)\times H^0(L'(D')) \to H^0(L(D)|_D)\times H^0(L'(D')|_D) \to H^0((L(D)|_D\otimes L'(D')|_D) \cong H^0((L(D)\otimes L'(D'))|_D) \cong H^0((L\otimes L')(D+D')|_D) \xrightarrow{\sim} H^0(\tilde{K}(D+D')|_D) \cong H^0(\mathcal{O}_D) \to k$.

Now if $\beta : \mathcal{B} \to \mathcal{A}^*$ is the linear map defined by the bilinear map $B : \mathcal{B}\times\mathcal{A} \to \mathcal{O}$, we check the skew-symmetry of the map $\mu = \beta\circ h : \mathcal{A} \to \mathcal{A}^*$. First note the intrinsic meaning of skew-symmetry here. The dual vector bundle $\mathcal{A}^*$ is $\mathcal{H}om_{\mathcal{O}}(\mathcal{A},\mathcal{O})$ and the standard natural identification $\mathcal{A} \xrightarrow{\sim} (\mathcal{A}^*)^*$ goes as follows: $a \mapsto (\gamma \mapsto \gamma(a))$ for local sections a of $\mathcal{A}$ and γ of $\mathcal{A}^*$. Then the linear map $\mu : \mathcal{A} \to \mathcal{A}^*$ has a dual linear map $\mu^* : \mathcal{A} \cong (\mathcal{A}^*)^* \to \mathcal{A}^*$, with $\mu^*(a)$ defined by the formula $\mu^*(a)(b) = (\mu(a))(b)$ for local sections b of $\mathcal{A}$. Skew-symmetry of μ means that $\mu^* = -\mu$, and this is equivalent to the condition that $(\mu(a))(b) = -(\mu(b))(a)$, in other words, that the bilinear form $(a,b) \mapsto (\mu(a))(b)$ is skew-symmetric (or alternating) in the usual sense.

Thus, let $R : \mathcal{A}\times\mathcal{A} \to \mathcal{O}$ be the bilinear form defined by $R(\sigma,\tau) = (\mu(\sigma))(\tau)$. We must check the skew-symmetry of this bilinear form and we can do that fibrewise. Thus we take $\sigma \in H^0(L(D)) \cong \mathcal{A}|_L$, calculate $\mu(\sigma) \in H^0(L(D))^*$, and apply it to $\tau \in H^0(L(D))$. We get $R(\sigma,\tau) = (\mu(\sigma))(\tau) = ((\beta\circ h)(\sigma))(\tau) = \beta(h(\sigma))(\tau) = B(\sigma|_D,\tau) = Res_D(<\sigma,\tau>)$, where $<\sigma,\tau>$ denotes $\phi(\sigma\otimes\iota^*(\tau)) \in H^0(\tilde{K}(D+D'))$, (as in [M2, p. 343]). But $\mathrm{Res}_D(<\tau,\sigma>) = Res_{D'}(\iota^*(<\tau,\sigma>)) = Res_{D'}(<\sigma,\tau>)$ and then $Res_{D'}(<\sigma,\tau>) = -Res_D(<\sigma,\tau>)$ by the residue theorem. So, $-Res_D(<\tau,\sigma>) = Res_D(<\sigma,\tau>)$ and skew-symmetry holds.

(iii) Now for our fixed $L_0 \in \Xi$, assume that D also satisfies $H^1(L_0(D-D')) = 0$. Note that the existence of such D does not present a problem. Indeed, what is clear is that if $Nm(L) = K_C$ and $h^0(L) = h^1(L) > 0$, then $\{p \in \tilde{C} \mid h^0(L(p)) = h^0(L)\}$ and $\{p \in \tilde{C} \mid h^0(L(-p')) = h^0(L)-1\}$ are nonempty open subsets of $\tilde{C}$. Hence there exists a point $p \in \tilde{C}$ outside of any specified finite subset of $\tilde{C}$ such that $h^0(L(p)) = h^0(L)$ and $h^0(L(-p')) = h^0(L)-1$; then for such p, $h^0(L(p-p')) = h^0(L)-1$. (There is a much more precise parity result that is known, but we do not want to assume that result here since we will deduce it in section 5(b) below.) Now $Nm(L(p-p')) = K_C$ and $h^1(L(p-p')) = h^0(L(p-p'))$ has gone down by 1 so we continue until we get the desired $D = p + \ldots$.

Now we claim that $\beta : \mathcal{B} \to \mathcal{A}^*$ is an isomorphism on the open neighborhood $U_2 = \{L \in P \mid \mathrm{H}^0(L(D-D')) = 0\}$ of L_0 in P. Note that $H^1(L(D-D')) = 0 \Rightarrow H^1(L(D)) = 0$ (from the short exact sequence $0 \to L(D-D') \to L(D) \to L(D)|_{D'} \to 0$), so $U_2 \subset U_1$.

It suffices to see that for each $L \in U_2$, the linear map $\beta_L : H^0(L(D)|_D) \to H^0(L(D))^*$ is an isomorphism, and β_L being an isomorphism is equivalent to the pairing B_L being perfect. But the restriction map $H^0(L(D)) \to H^0(L(D)|_{D'})$ is an isomorphism by the hypothesis (since the kernel of this mapping is $H^0(L(D-D')) = 0$, and the 2 vector spaces have the same dimension). Thus, the analysis of the pairing B_L on $H^0(L(D)|D) \times H^0(L(D))$ is reduced to the case of the pairing $H^0(L(D)|_D) \times H^0(L(D)|_{D'}) \cong H^0(L(D)|_D) \times H^0(L'(D')|_D) \to k$, which is easily seen to be perfect by checking pointwise on D.

(iv) Finally, assume that the divisor D consists of exactly $n = h^0(L_0)$ points (with the previous assumptions holding as well) and take any trivialization of $\mathcal{A}$ on an open neighborhood U of L_0 in $U_2 \subset P$. Then, in terms of the basis of sections $s_1, ..., s_n$ for $\mathcal{A}$ on U and the Kronecker dual basis of sections $t_1, ..., t_n$ of $\mathcal{A}^*$, let M be the matrix representing μ. That is, M has i, j entry $f_{ij} = \mu(s_j)(s_i)$. Then $M = (f_{ij})$ is an $n \times n$ matrix of regular functions on U and this matrix is skew-symmetric since the corresponding linear map μ is skew-symmetric. Now we address Prym-Petri and Abel. These follow readily from the corresponding well-established properties in the Jacobian case. Indeed, for (2), μ has the same family of kernels on U as h, only now the matrices representing the linear maps μ_L are skew-symmetric.

For (1), we will express the first order variation of $\{\mu_L\}$ at L_0 in terms of the first order variation of $\{h_L\}$ and the map β_{L_0} itself. Namely, consider Taylor expansions $h = h_1 + h_2 + \dots$ and $\beta = \beta_0 + \beta_1 + \dots$ for h and β around L_0. (Just take arbitrary local trivializations for $\mathcal{A}$ and $\mathcal{B}$, and represent $h : \mathcal{A} \to \mathcal{B}$ and $\beta : \mathcal{B} \to \mathcal{A}^*$ by matrices. The term $h_0 = 0$ since h has been arranged to vanish on the fibre at L_0.) Then the Taylor expansion for the composition is given by matrix multiplication: $\mu = \beta \circ h = (\beta_0 + \beta_1 + \dots) \circ (h_1 + h_2 + \dots) = \beta_0 \circ h_1 + (\beta_0 \circ h_2 + \beta_1 \circ h_1) + \dots$. All we need to identify is the linear term $\beta_0 \circ h_1$. We assume from the Jacobian case (for $\tilde{C}$) that h_1 matches up with the restriction to $T_0(P) \subset T_0(\tilde{J})$ of the map $T_0(\tilde{J}) \to Hom(H^0(L_0), \mathrm{H}^1(L_0))$ corresponding to the Petri map (in one of the several ways the Petri map and its dual can be expressed). Recall here that since h vanishes at L_0, we have isomorphisms $H^0(L_0) \xrightarrow{\sim} H^0(L_0(D)) = \mathcal{A}_{L_0}$ and $\mathcal{B}_{L_0} = H^0(L_0(D)|_D) \xrightarrow{\sim} H^1(L_0)$.

Now we know what the map $\beta_0 : H^0(L_0(D)|_D) \xrightarrow{\sim} H^0(L_0(D))^*$ is, but we have to check that the induced isomorphism $H^1(L_0) \cong H^0(L_0)^*$ is the one we want. More generally consider the similar situation for any $L \in P$:

$$\begin{array}{ccc} H^0(L(D)|_D) & \xrightarrow{\beta_L} & H^0(L(D))^* \\ \downarrow & & \downarrow \\ H^1(L) & & H^0(L)^* \end{array}$$

We will check that β_L induces the natural isomorphism $H^1(L) \xrightarrow{\sim} H^0(L)^*$ defined successively by $H^1(L) \cong H^0(\tilde{K} \otimes L^*)^* \cong H^0(L')^* \cong H^0(L)^*$. Thus, look at the 2 pairings: $B_L : H^0(L(D)|_D) \times H^0(L(D)) \to k$ and $C : H^1(L) \times H^0(L) \to k$. Take $s \in H^0(L(D)|_D)$

and its image $t \in H^1(L)$, and take $u \in H^0(L)$ and its image $\tau \in H^0(L(D))$. We claim that $B_L(s,\tau) = C(t,u)$. Here, $B_L(s,\tau)$ is defined as before to be $r_D(\phi(s \otimes \iota^*(\tau)|_D))$, and $C(t,u)$ is calculated as follows. Suppose $u \leftrightarrow \iota^*(u) = u' \leftrightarrow \hat{u}$ under the isomorphisms $H^0(L) \cong H^0(L') \cong H^0(\tilde{K} \otimes L^*)$. Then $C(t,u)$ is the scalar corresponding to the element $t \cup \hat{u} \in H^1(\tilde{K})$ under the fundamental isomorphism $H^1(\tilde{K}) \xrightarrow{\sim} k$.

Now set $N = \tilde{K} \otimes L^*$. There is the multiplication pairing $H^0(L(D)|_D) \times H^0(N(D')) \to H^0(L \otimes N(D+D')|_D)$, but we can canonically identify $N(D+D')|_D$ with $N(D)|_D$ since D and D' are disjoint effective divisors. Then what we need boils down to the fact that the isomorphism $H^1(\tilde{K}) \xrightarrow{\sim} k$ is induced by the residue map $H^0(\tilde{K}(D)|_D) \to k$, and the commutativity of the following diagram:

$$\begin{array}{ccc} H^0(L(D)|_D) \times H^0(N) & \longrightarrow & H^0(L \otimes N(D)|_D) \\ \big\downarrow & & \big\downarrow \\ H^1(L) \times H^0(N) & \longrightarrow & H^1(L \otimes N) \end{array}$$

That is, the upper multiplication pairing involving principal parts of L induces the lower cup product pairing on cohomology; see [K2, §3,6, esp. p. 43].

Now we have formed the map $T_0(P) \to Hom(H^0(L), H^0(L)^*) \cong \{$bilinear forms on $H^0(L) \times H^0(L)\}$. However,as Mumford indicated [M2, p. 343] (in his notation, that $\omega_{\bar{ij}}$ is skew-symmetric), this map from $T_0(P)$ actually goes to $\{$skew-symmetric bilinear forms on $H^0(L) \times H^0(L)\}$, and thus agrees with the Prym-Petri map. Q.E.D.

Example: stable double points of Ξ. Recall that a double point L of Ξ is called stable (w.r.t. the given étale double cover $\pi : \tilde{C} \to C$) if $h^0(\tilde{C}, L) = 4$. Then

$$M = \begin{bmatrix} 0 & x & a & b \\ * & 0 & c & d \\ * & * & 0 & y \\ * & * & * & 0 \end{bmatrix} \quad \text{and} \quad Pf(M) = xy - ad + bc,$$

where x, y, a, b, c, d are regular functions vanishing at L, so the quadratic form defining the quadric tangent cone $C_L(\Xi)$ has rank ≤ 6.

6. Remarks on global aspects of the local Pfaffian structure defining a Prym theta divisor

If $\mu : \mathcal{A} \to \mathcal{A}^*$ is the skew-symmetric vector bundle homomorphism constructed above on P, then its Pfaffian $Pf(\mu)$ defines the Prym theta divisor Ξ in $U = \{L \in P \mid \mathrm{H}^0(L(D - D')) = 0\}$. Here the Pfaffian line bundle of μ is $\Lambda^n(\mathcal{A}^*)$, where $n = deg(D)$ is the rank of $\mathcal{A}$, and $Pf(\mu)$ is a section whose square is the section of $(\Lambda^n(\mathcal{A}^*))^{\otimes 2}$ corresponding to the homomorphism $\Lambda^n(\mu) : \Lambda^n(\mathcal{A}) \to \Lambda^n(\mathcal{A}^*)$ induced by μ on the top exterior power. However, $Pf(\mu)$ vanishes on the complement of U as well. In other words, the complex $(\mathcal{A} \xrightarrow{\mu} \mathcal{A}^*)$ on P could be viewed as a global "Pfaffian structure" on P defining in fact $\Xi \cup (\Xi + \tau)$, where τ is the point $\mathcal{O}_{\tilde{C}}(D' - D)$ of P_0 (the abelian variety model of P inside the abelian variety model $\tilde{J}_0 = Pic^0(\tilde{C})$ of the Jacobian of the double cover). Thus we return to the original complex $(\mathcal{A} \xrightarrow{h} \mathcal{B})$ on P, and then we obtain the following result.

Proposition: Assume that the effective divisor D on $\tilde{C}$ has degree at least $\tilde{g}$, consists of distinct points, and that D and its conjugate D' are disjoint. Consider the previously constructed homomorphism $h : \mathcal{A} \to \mathcal{B}$ as a 2-term complex $(\mathcal{A} \to \mathcal{B})$ of vector bundles on P. Then there exists a homomorphism δ from the complex $(\mathcal{A} \to \mathcal{B})$ to the complex $(\mathcal{B}^* \to \mathcal{A}^*)$ such that δ is skew-symmetric and is a quasi-isomorphism on all of P.

Proof: With $\beta : \mathcal{B} \to \mathcal{A}^*$ as before, set $\alpha = -\beta^* : \mathcal{A} \to \mathcal{B}^*$. Then, the following diagram commutes

$$\begin{tikzcd} \mathcal{B} \arrow[r, "\beta"] & \mathcal{A}^* \\ \mathcal{A} \arrow[u, "h"] \arrow[r, "\alpha"] & \mathcal{B}^* \arrow[u, "h^*"'] \end{tikzcd}$$

since the composition $\mu = \beta \circ h : \mathcal{A} \to \mathcal{A}^*$ is a skew-symmetric vector bundle homomorphism by the previous calculations. In other words, we know that $(\beta \circ h)^* = -\beta \circ h$, so $h^* \circ \beta^* = -\beta \circ h$ and hence $h^* \circ \alpha = \beta \circ h$ as desired.

Thus $\delta = (\alpha, \beta)$ is a homomorphism of complexes from $(\mathcal{A} \to \mathcal{B})$ to $(\mathcal{B}^* \to \mathcal{A}^*)$. By skew-symmetry of δ we mean simply that $\alpha = -\beta^*$ (and $\beta = -\alpha^*$), which holds by construction.

It remains to check that δ is a quasi-isomorphism of complexes, i.e. induces an isomorphism on the cohomology sheaves. Since the kernel sheaves of the 2-term complexes are 0, all we have to check is that β induces an isomorphism on the cokernel sheaves. For this, it is enough to check pointwise that both α_L and β_L induce isomorphisms on the $0^{\underline{th}}$ and $1^{\underline{st}}$ cohomology vector spaces at L, i.e. that the induced maps $c_0 : H^0(L) \to H^1(L)^*$ and $c_1 : H^1(L) \to H^0(L)^*$ are isomorphisms. (Indeed, if $c : \mathcal{S} \to \mathcal{T}$ is the map on the cokernel sheaves, then the induced map $\mathcal{S}|_L \to \mathcal{T}|_L$ on the fibres at L is c_1 and the induced map $\mathcal{T}or_1(\mathcal{S}, k_L) \to \mathcal{T}or_1(\mathcal{T}, k_L)$ is c_0. So by Nakayama's Lemma, if c_1 is surjective for every L, then $coker(c) = 0$, and after that, if c_0 is surjective and c_1 is injective, then $ker(c) = 0$.) But $c_1 = -c_0^*$, and the source and target have the same dimension, so it suffices to show that c_0 is injective. Thus, suppose that $c_0(\tau) = 0$ for $\tau \in H^0(L) \subset H^0(L(D)) = \mathcal{A}|_L$. Then $-B_L(\cdot, \tau) = 0$ in $H^0(L(D)|_D)^* = \mathcal{B}^*|_L$, so $\iota^*(\tau)$ must lie in $H^0(L'(-D))$ and hence τ must lie in $H^0(L(-D')) = 0$ (since $deg(L(-D')) < 0$). Q.E.D.

The result just presented is our analysis of the exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{S} \to 0$ on the Prym variety. We were unaware of the general result, Theorem 9.2 (skew-symmetric version) of [E-P-W2]. In the terminology of [E-P-W2, §9], the cokernel $\mathcal{S}$ is a "skew-symmetric sheaf" on P since our Proposition above produces the resolution structure in their Thm. 9.2, part (b) (without twisting by a line bundle). This sheaf $\mathcal{S}$, whose fibre at L is $H^1(L)$, has the structure of a rank 2 vector bundle on the smooth points Ξ_{sm} of the Prym theta divisor, and in fact determines the étale double cover $\pi : \tilde{C} \to C$, as long as the singular locus of Ξ has dimension less than g-5; see [S-V3] and [N].

Although the main constructions for Prym varieties in this paper are quite similar to those in [M1, pp. 182-184, 186-188], they are not exactly the same. Let us spell out the relation. Mumford uses a divisor on the base curve C and, in the notation of [F-P, p. 94], forms a vector bundle $\mathcal{V}$ on P with a nondegenerate $\mathcal{O}$-valued quadratic form, and

2 maximal isotropic subbundles $\mathcal{U}$ and $\mathcal{W}$ of $\mathcal{V}$. Let us write the pullback divisor on $\tilde{C}$ in the form $D + D'$ and use $\overline{D} = Nm(D)$ to denote the divisor on C. Recall that for $L \in P$, the direct image of the line bundle L on $\tilde{C}$ is the rank 2 vector bundle $\mathcal{E} = \pi_*(L)$ on C, with fibres $\mathcal{E}_{\bar{p}} = L_p \oplus L_{p'} \cong L_p \oplus L'_p$, where $\pi^{-1}(\bar{p}) = \{p, p'\}$. Then the fibres of $\mathcal{V}$, $\mathcal{W}$, and $\mathcal{U}$ at L are:

$$\mathcal{V}_L = \mathcal{E}(\overline{D})/\mathcal{E}(-\overline{D}) \cong L(D)/L(-D) \oplus L'(D)/L'(-D)$$

$$\mathcal{W}_L = H^0(\mathcal{E}(\overline{D})) \text{ and } \mathcal{U}_L \cong L/L(-D) \oplus L'/L'(-D).$$

Our vector bundle $\mathcal{A}$ has $\mathcal{A}_L = H^0(L(D))$, and is half the size of $\mathcal{W}$ (and our $\mathcal{B}$ has half the size of $\mathcal{U}$). The vector bundle $\mathcal{A}$ has the skew-symmetric bilinear residue form R defined on it. Note, by the way, that one could proceed as in [F-P, pp. 71-72] to consider $\mathcal{A} \oplus \mathcal{A}^*$ with its canonical symmetric(!) form so that the graph $\{(a, R(a, \cdot))\}$ is a maximal isotropic subbundle, and $\mathcal{A} \oplus 0$ is another maximal isotropic subbundle. Instead, we can map $\mathcal{A}$ to the bundle $\mathcal{V}$ in Mumford's construction. To see the relationship between the alternating form on $\mathcal{A}$ and the (nondegenerate) quadratic form on $\mathcal{V}$, look at the fibre $\mathcal{V}_L$. Since $L' \cong \tilde{K} \otimes L^*$, we have $L(D)/L(-D) \oplus (\tilde{K} \otimes L^*)(D)/(\tilde{K} \otimes L^*)(-D)$ and Mumford pairs from $L(D) \times (\tilde{K} \otimes L^*(D))$ to $\tilde{K}(2D)$ and takes the residue. It had been noted in [E-P-W2, §2], that the data present in the orthogonal bundle setup ($\mathcal{U}, \mathcal{W} \subset \mathcal{V}$) could be used locally, after a certain choice, to obtain a skew-symmetric map; namely, the natural map $\mathcal{U} \to \mathcal{W}^*$ preceded by a suitable local isomorphism from $\mathcal{W}$ to $\mathcal{U}$ becomes (locally) a skew-symmetric map from $\mathcal{W}$ to $\mathcal{W}^*$. Independently, we found a direct local construction without any further choices. In essence, we have gone back to the explicit residue formula embedded in Mumford's original construction and emphasized the skew-symmetry property of the residue formula in the Prym variety case.

7. Applications of the local Pfaffian structure defining a Prym theta divisor

(a) Some standard features

- $\tilde{\Theta}|_P = 2\Xi$ since the determinant of a skew symmetric matrix is the square of the Pfaffian.
- $h^0(L)$ is even for all $L \in P$ since for a family of skew-symmetric matrices, the kernel dimension has constant parity. [In order for this property of P not to appear tautological, suppose here that P is defined as a connected component of $Nm^{-1}(K_C) \subset \tilde{J}$ that is not entirely contained in $\tilde{\Theta}$.]
- The parity of $h^0(L)$ is opposite on the 2 connected components of $Nm^{-1}(K_C)$. See part (b) below.
- Mumford's Pfaffian of linear terms for $L \in \Xi$ either provides an equation for the tangent cone $C_L(\Xi)$ or else is identically 0. See part (c) below.
- Two types of singularities are possible on the Prym theta divisor since a point $L \in \Xi$ is singular if and only if it is pulled back from a singular point of the locus of singular skew-symmetric matrices or is a singular point of the map to the skew-symmetric matrices. [Recall that $L \in \Xi$ is a stable singular point if $h^0(\tilde{C}, L) \geq 4$, and is an exceptional singular point if there exists a line bundle M on C with $h^0(C, M) \geq 2$ and an effective divisor B

on $\tilde{C}$ such that $L \cong \pi^*(M)(B)$. Note that the 2 types of singularities are defined in terms of the étale double cover $\pi : \tilde{C} \to C$ and are not mutually exclusive.]

In this connection, note the following hierarchy of conditions on the germ of an inducing map $M : (P, L) \to Alt(n)$ for $\Xi \subset P$ around a point $L \in \Xi$. (i) the Prym-Petri map $\Lambda^2(H^0(L)) \to H^0(C, K_C(\eta))$ is injective $\Leftrightarrow$ (ii) M is submersive $\Rightarrow$ (iii) M is transverse to the map $\{(A, [v]) \in Alt(n) \times \mathbb{P}^{n-1} \mid Av = 0\} \to Alt(n) \Leftrightarrow$ (iv) X is smooth along $|L|$ (given that X is known to be purely of the expected dimension $g-1$) $\Rightarrow$ (v) the tangent cone $C_L(\Xi)$ is defined by Mumford's Pfaffian of linear terms. The implication (iv) $\Rightarrow$ (v) was proved in [S-V2, Thm. 2.1].

(b) The Wirtinger-Mumford parity results

We will give an analysis of the fundamental parity flipping result that Mumford proved in [M1] (and attributed to Wirtinger in the classical, analytic case). A nice direct argument can be found in [T, proof of Lemma 1.6, p. 955].

Theorem (Wirtinger and Mumford): Suppose that $L \in Pic^{2g-2}(\tilde{C})$ is "precanonical", i.e. $Nm(L) = K_C \in Pic^{2g-2}(C)$. Then for any point $p \in \tilde{C}$, the parity of $h^0(L(p-p'))$ is opposite that of $h^0(L)$. More precisely, either
(a) $h^0(L(p-p') = h^0(L) - 1$ (in case $|L| \neq \phi$ and p' is not a base point of $|L|$), or else
(b) $h^0(L(p-p') = h^0(L) + 1$ (in case $|L| = \phi$ or p' is a base point of $|L|$).

Proof: We set up the local Pfaffian structure around the point L of $Pic^{2g-2}(\tilde{C})$ and then use a closely related Pfaffian stucture around $L(p-p')$. Assume that the pair (D, D') is adapted to L, i.e. $H^0(L(D-D')) = 0$. (The existence of such a pair was argued in part (iii) of the proof in section 5. In the terminology of [K2, p. 34], a line bundle L is called "adapted" to a pair (D, E) of effective divisors if $deg(L) + deg(D) - deg(E) = g-1$ and $H^0(L(D-E)) = 0$. It follows readily [K2, Lemma 5.1] that $H^1(L(D-E)) = 0$ and the restriction map $H^0(L(D)) \to H^0(L(D)|_E)$ is an isomorphism. Kempf's monograph [K2] then goes on to show in detail how the pair (D, E) helps analyze the linear map $H^0(L(D)) \to H^0(L(D)|_D)$, which is a complex computing the cohomology of L.)

Then note that the pair $(D+p', D'+p)$ is adapted to $L(p-p')$ since $L(p-p')((D+p')-(D'+p)) \cong L(D-D')$. Now we know that $H^0(L)$ is realized as the kernel of the skew-symmetric linear map $\mu_L : H^0(L(D)) \to H^0(L(D))^*$. And likewise $H^0(L(p-p'))$ is realized as the kernel of the skew-symmetric linear map $\mu_{L(p-p')} : H^0(L(D+p')) \to H^0(L(D+p'))^*$. Note that the first vector space $H^0(L(D))$ has dimension $h^0(L(D)) = deg(D)$ and the second vector space $H^0(L(D+p'))$ has dimension $h^0(L(D+p')) = deg(D+p') = deg(D)+1$. Also, the first vector space injects naturally (left vertical arrow) into the second vector space so that the following diagram commutes:

$$\begin{array}{ccc} H^0(L(D+p')) & \xrightarrow{\mu_{L(p-p')}} & H^0(L(D+p'))^* \\ \uparrow & & \downarrow \\ H^0(L(D)) & \xrightarrow{\mu_L} & H^0(L(D))^* \end{array}$$

If we take compatible bases for the 2 vector spaces, the larger skew-symmetric matrix M_2 (for the upper horizontal map $\mu_{L(p-p')}$) consists of the smaller skew-symmetric matrix

M_1 (for the lower horizontal map μ_L) plus an additional row and column. The ranks of M_1 and M_2 have the same parity (since both ranks are even), and hence the coranks of M_1 and M_2 have opposite parity. Q.E.D.

(c) The role of higher order terms in the Pfaffian structure

Suppose that the "Riemann singularity theorem fails" for $L \in \Xi$, i.e. that $Pf(M_1)$, Mumford's Pfaffian of the skew-symmetric matrix of linear terms, is identically 0 on the tangent space $T_L(P)$. Then the multiplicity of Ξ at L is greater than $\frac{1}{2}h^0(L)$. In [S-V5] it was shown that there exists a unique line bundle $\mathcal{M}$ on C with $h^0(C, \mathcal{M}) > \frac{1}{2}h^0(L)$ and effective divisor B on $\tilde{C}$ with $B \cap B' = \phi$ such that $L \cong \pi^*(\mathcal{M})(B)$. We called such a singular point $L \in \Xi$ "very exceptional", and $(\mathcal{M}, B)$ a "Shokurov pair" for L (influenced by [S, Lemma 5.7, p. 121]). We noted that $mult_L(\Xi) \geq h^0(\mathcal{M})$, and asked whether equality must hold. Casalaina-Martin [CM] has recently established the equality and we will just indicate the interpretation of his result in terms of the skew symmetric matrix M describing Ξ around L. Take a basis for $H^0(L)$ starting with a basis for $\pi^*(H^0(\mathcal{M}))$ and look at the block form of M. Then

$$M = \begin{bmatrix} Q & | & \Lambda \\ - & & - \\ * & | & \kappa \end{bmatrix} + \dots \quad \text{and} \quad M_1 = \begin{bmatrix} 0 & | & \Lambda \\ - & & - \\ * & | & \kappa \end{bmatrix}.$$

Here, if we set $m = h^0(C, \mathcal{M})$ and $n = h^0(\tilde{C}, L)$, then Q is an $m \times m$ skew-symmetric block of quadratic forms, Λ is an $m \times (n-m)$ block of linear forms κ is an $(n-m) \times (n-m)$ skew-symmetric block of linear forms, and the dots indicate matrices of appropriate higher order terms. The matrix Q can be modified by the addition of $\Lambda F - F^t\Lambda^t$ for any $(n-m) \times m$ matrix F of linear forms (appearing in the linear part of a change of local trivialization of the vector bundle $\mathcal{A}$ around L). In each row and in each column of Λ, the linear forms are linearly independent (by [S-V5, Lemma 2.3 iii)], using the maximality assumption on the pair $(\mathcal{M}, B)$).

Now Casalaina-Martin's result is that the order of the first nonvanishing term in $Pf(M)$ is exactly $m = h^0(C, \mathcal{M})$ ($> \frac{1}{2}n$, where $n = h^0(\tilde{C}, L)$). As a consequence, whenever Mumford's homogeneous polynomial $Pf(M_1)$ of degree $\frac{1}{2}h^0(\tilde{C}, L)$ is identically 0, the homogeneous polynomial $Pf_m(M) = Pf(\begin{bmatrix} Q & | & \Lambda \\ - & & - \\ * & | & 0 \end{bmatrix})$ of larger degree provides the equation for the tangent cone $C_L(\Xi)$.

To check this derivation, consider the full expansion formula for $det(M)$ in which each term is $\pm$ a product of entries, one from each row and column. For any single product, consider the entries taken from the last $n-m$ columns and let r be the number of these entries that lie in the first m rows. Then exactly $m-r$ entries in the product must come from the upper left block, and the remaining r entries in the product must be chosen from the lower left block. Therefore the order of such a product is $\geq 1\cdot(n-m)+2\cdot(m-r)+1\cdot r = n+m-r$, But then $n+m-r = 2m+(n-m-r) \geq 2m$ since obviously $r \leq n-m$. Thus every term in $det(M)$ has order at least $2m$ and only terms with $r = n-m$ can have order exactly $2m$. Thus, the determinant of the matrix $\begin{bmatrix} Q & \Lambda \\ -\Lambda^t & 0 \end{bmatrix}$ (with homogeneous entries and lower right block 0) contributes precisely all the homogeneous terms of order exactly $2m$ in $det(M)$.

Example: the conic bundle structure of a cubic threefold

For a smooth cubic threefold $W \subset \mathbb{P}^4$ and a generic line $\ell \subset W$, consider the conic bundle structure $Bl_\ell W \longrightarrow \mathbb{P}^2$ arising from projection of W from the line. Let $C \subset \mathbb{P}^2$ be the quintic discriminant curve and let $\pi : \tilde{C} \to C$ be the étale double cover whose points are the lines occurring in reducible fibres of the conic bundle structure. It is known that the associated Prym variety has a unique singular point L on its theta divisor and the cubic threefold W appears as the projectivized tangent cone at this point. (See [M2, p. 348], [B], cf. [S-V2, §5] and the references there, and also cf. [CM-F].) For the Pfaffian structure around L,

$$M = \begin{bmatrix} 0 & Q_2 & Q_1 & x_0 \\ * & 0 & Q_0 & x_1 \\ * & * & 0 & x_2 \\ * & * & * & 0 \end{bmatrix} + \dots \quad \text{and} \quad Pf_3(M) = x_0Q_0 - x_1Q_1 + x_2Q_2$$

is an equation for the cubic threefold $W \subset \mathbb{P}^4$ in a form that displays the conic bundle structure for projection of W from the line ℓ defined by the vanishing of x_0, x_1, x_2. This example shows that the local Pfaffian structure is not an invariant of the principally polarized abelian variety (P, Ξ) but depends on the étale double cover $\pi : \tilde{C} \to C$.

8. References

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