Sponsored by: UGA Math Department and UGA Math Club

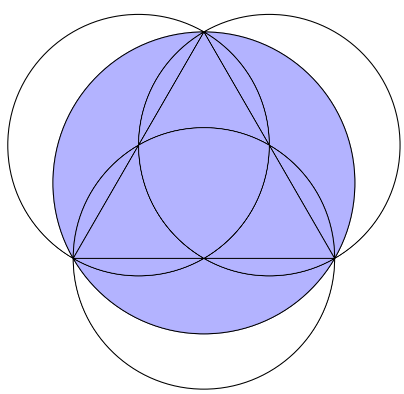
TEAM ROUND / 1 HOUR / 210 POINTS
October 22, 2022

WITH SOLUTIONS

Problem 1 (Going in circles)**.** What is the smallest radius r so that 3 disks of raidus r can completely cover a disk of raidus 1?

Answer. $\frac{\sqrt{3}}{2}$

Solution. In order for 3 disks of radius r to cover a disk of radius 1, they must at least cover the boundary circle of radius 1. The largest part of a circle that a disk can cover occurs when a diameter of the disk is a chord of the circle. For 3 disks to cover the entire circle, make their diameters form an inscribed equilateral triangle. The radius r is then half the side length of the inscribed equilateral triangle: $\frac{\sqrt{3}}{2}$. These three discs cover the entire circle of radius 1, since the radius is large enough to reach the center.



Problem 2 (Making a difference)**.** Suppose a, b, c, and d are distinct positive integers satisfying

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}<1.$$

Define the difference D as

$$D=1-\frac{1}{a}-\frac{1}{b}-\frac{1}{c}-\frac{1}{d},$$

and write $D=r/s$ in lowest terms. If a, b, c, and d are chosen so that D is as small as possible, what is $r+s$?

Answer. The smallest possible value of D is $1/1806$, and so $r+s=1807$.

Solution. To set up for the solution, we introduce the term *unit fraction* to mean a fraction of the form $1/n$, where n is a positive integer. Thus, D is the smallest error one can make when approximating 1 from below by a sum of four unit fractions.

For two unit fractions, the situation is easy: The best approximation to 1 from below is $5/6$, which is achieved by $\frac{1}{2}+\frac{1}{3}$. (We leave it to you to verify this.) Now, notice that if we have an approximation to 1 from below by n unit fractions, with an error of exactly $\frac{1}{d}$, appending the fraction $\frac{1}{d+1}$ to our list gives a new approximation from below involving $n+1$ unit fractions where the error is $\frac{1}{d(d+1)}$. Since we have an approximation with error $\frac{1}{6}$ with $d=2$, we have one with error $\frac{1}{6(7)}=\frac{1}{42}$ with $d=3$ and one with error $\frac{1}{42(43)}=\frac{1}{1806}$ with $d=4$. Keeping track of the fractions involved, we see that $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}$ achieves this bound.

It is believable but far from obvious that we cannot get closer than $\frac{1}{1806}$. The proof we give below requires some numerical legwork supplemented by the following CLAIM. Suppose $A<B$ are positive real numbers. Suppose also that we have a lower bound on A, say $A\geq C$ (where C is also positive) and a lower bound on the product AB, say $AB\geq P$ (with P positive too). Then the largest $1/A+1/B$ can be occurs when A and AB are as small as allowed (that is, $A=C$ and $B=P/C$). In other words, $1/A+1/B\leq 1/C+C/P$.

To prove the CLAIM, fix the product AB, say $AB=M$, so that $B=M/A$. We first argue that $1/A+1/B\leq 1/C+C/M$. Indeed, if we subtract the left-hand side from the right we get $(A-C)(1/AC-1/M)$. Here $A-C\geq 0$ (by assumption). Furthermore, $C\leq A<B$, and so $AC<AB=M$. Hence,

$1/AC - 1/M > 0$. Recalling that $M \geq P$, we conclude that $1/A + 1/B \leq 1/C + C/M \leq 1/C + C/P$, proving the CLAIM. Moreover, tracing through the argument shows equality holds if and only if $A = C$ and $B = P/C$.

Before showing 1/1806 is optimal for 4 fractions, let us show 1/42 is optimal for 3 fractions. This is not strictly necessary but illustrates the methods.

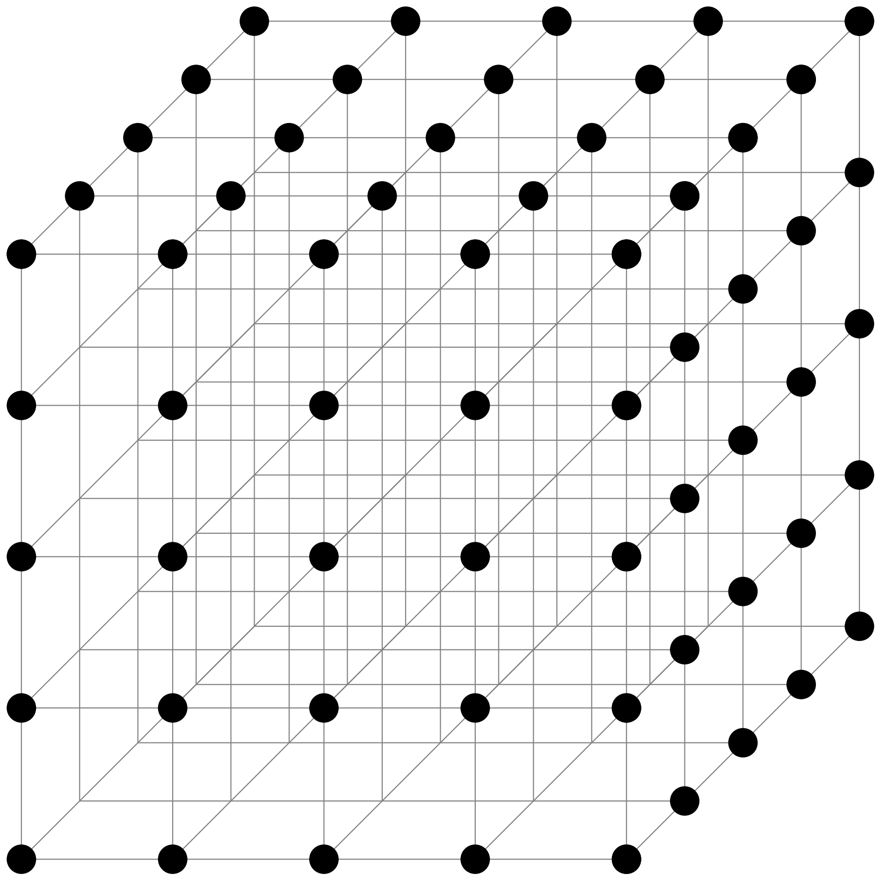
Suppose $1/a + 1/b + 1/c < 1$ is smaller than 1 but within 1/42 of 1. Observe that the difference $1 - 1/a - 1/b - 1/c$ is positive and can be written as a fraction with denominator abc. Hence, for this difference to be at most 1/42, it must be that $abc \geq 42$. To continue, assume WLOG that $a < b < c$. We must have $a = 2$. Otherwise, $a \geq 3$, and $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5}$. But a crude numerical approximation of the RHS shows that it is smaller than 0.8, so there is no way $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ can be within 1/42 of 1. So we may assume $a = 2$. Then $bc \geq 21$ and $b \geq 3$. By the above CLAIM, $\frac{1}{2} + \frac{1}{b} + \frac{1}{c} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} = \frac{41}{42}$. This shows that 1/42 is optimal with three fractions and is achieved only when $a = 2$, $b = 3$, and $c = 7$.

With the warm up out of the way, we turn to four unit fractions. Suppose $1/a + 1/b + 1/c + 1/d$ is smaller than 1 but within 1/1806 of 1. Analogously to the above, we assume $a < b < c < d$. Then (arguing as above) we must have $abcd \geq 1806$. We must also have $a = 2$, since otherwise $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = 19/20$, which is too small.

Continuing, we notice that b must be $3, 4$, or 5. Indeed, if $b \geq 6$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{2} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$. The upper bound here is smaller than $\frac{1}{2} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$. But it cannot be within 1/1806 of 1, since $2 \cdot 6 \cdot 7 \cdot 8 < 1806$. So if $b \geq 6$, then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ is again too small.

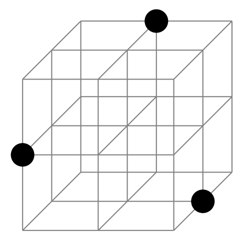
Suppose $b = 5$. Then $cd \geq 1806/ab = 1806/10$, and hence $cd \geq 181$. Also, $c \geq 6$. By the CLAIM, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{181/6} < \frac{1}{2} + \frac{1}{5} + \frac{1}{6} + \frac{1}{30} = \frac{9}{10}$, which is far too small. Similarly, if $b = 4$, then $cd \geq 1806/8$, so that $cd \geq 226$. Also, $c \geq 5$. But then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{226/5} < \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{45} = \frac{35}{36}$. Again, this is far too small. So $b = 3$. Now in order for $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ to be smaller than 1, it must be that $c \geq 7$. But also $cd \geq 1806/6 = 301$. Hence, by the CLAIM once more, $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \leq \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{301/7} = \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} = \frac{1805}{1806}$. This implies that 1/1806 is optimal and only achieved when $a = 2, b = 3, c = 7$, and $d = 43$.

Problem 3 (Descent into madness)**.** How many equilateral triangles can be formed using the integer points which lie in the cube $[0,4] \times [0,4] \times [0,4]$? Note: The integer points on the surface of the cube are also included for a total of 125 integer points.



Answer. 1264

Solution. The intended practical approach to solving this question is to observe that you can generate equilateral triangles by choosing points symmetrically with respect to a vertex of the cube (say all a knights move away from that vertex) e.g. in the $2 \times 2 \times 2$ cube below.



There is just one additional case to find: One type of equilateral triangle is not so aligned, for example the equilateral triangle formed by $A = (0, 3, 0)$, $B = (4, 4, 1)$, and $C = (3, 0, 0)$. This exception arises because it is possible to write 18 as the sum of 3 squares in two different ways. Careful counting then yields the solution of 1264 triangles.

What follows below is a more rigorous approach to the problem.

We can detect if three points form an equilateral triangle by checking if the distance between each pair of points is the same. Given points two points $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$, the distance between them is

$$\sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2}. \qquad (*)$$

Now suppose that $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, and $C = (c_1, c_2, c_3)$ form an equilateral triangle, and define $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ to be the vector from A to B and the vector from B to C respectively, i.e.

$$\vec{u} = (b_1 - a_1, b_2 - a_2, b_3 - a_3), \qquad \vec{v} = (c_1 - b_1, c_2 - b_2, c_3 - b_3).$$

If we similarly define the vector $\vec{w}$ from C to A, then we can observe $\vec{u} + \vec{v} + \vec{w} = \vec{0}$ so $\vec{w} = -(\vec{u} + \vec{v})$.

Because A, B, and C form an equilateral triangle, the lengths of $\vec{u}$, $\vec{v}$, and $\vec{w}$ must be equal, and indeed if ℓ is the sidelength of the triangle then

$$\ell^2 = u_1^2 + u_2^2 + u_3^2 = v_1^2 + v_2^2 + v_3^2 = (u_1 + v_1)^2 + (u_2 + v_2)^2 + (u_3 + v_3)^2.$$

Expanding out the right hand side, we find ℓ^2 must be even as

$$\ell^2 = 2\ell^2 + 2(u_1v_1 + u_2v_2 + u_3v_3).$$

Now we catalog all of the possibilities for vectors $\vec{u}$ in the $4 \times 4 \times 4$ cube where $u_1^2 + u_2^2 + u_3^2 = \ell^2$ is even. The table below is grouped in terms of the largest magnitude component of u.

$\{\lvert u_1\rvert, \lvert u_2\rvert, \lvert u_3\rvert\}$	ℓ^2
$\{1,0,1\}$	2
$\{2,0,0\}$	4
$\{2,0,2\}$	8
$\{2,1,1\}$	6
$\{2,2,2\}$	12
$\{3,0,1\}$	10
$\{3,0,3\}$	18
$\{3,1,2\}$	14
$\{3,2,3\}$	22
$\{4,0,0\}$	16
$\{4,0,2\}$	20
$\{4,0,4\}$	32
$\{4,1,1\}$	18
$\{4,1,3\}$	26
$\{4,2,2\}$	24
$\{4,2,4\}$	36
$\{4,3,3\}$	34
$\{4,4,4\}$	48

Note that every ℓ^2 value is unique except 18 which can be generated by either $\{3,0,3\}$ or $\{4,1,1\}$. This means that if we hope to have equilateral triangles of sidelength $\ell \neq \sqrt{18}$, they can only be generated by vectors with the same magnitude coordinates. To take a specific example, if we want a cube of sidelength $\sqrt{6}$ it can only be formed with three points A, B, and C such that the coordinates of the vectors between any two of them take the form $(2,1,1)$ in some order and with some signs.

Case 1: Three vectors of the same coordinate magnitudes. We will reduce the above table to the list where it's actually possible to use three vectors of the same coordinate magnitudes to add to $\vec{0}$ (recalling we need $\vec{u} + \vec{v} + \vec{w} = \vec{0}$).

For example, the $\{2,1,1\}$ option remains in the table below because there are choices for $\vec{u}$, $\vec{v}$, and $\vec{w}$ with these coordinate magnitudes such that $\vec{u} + \vec{v} + \vec{w} = \vec{0}$, e.g.

$$\vec{u} = (2,1,1), \quad \vec{v} = (-1,-2,1), \quad \vec{u} = (-1,1,-2).$$

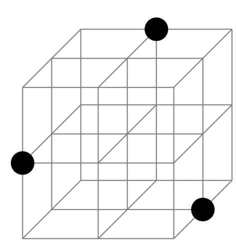
On the other hand $\{2,0,0\}$ has been removed from the table because no such valid choice for $\vec{u}$, $\vec{v}$, and $\vec{w}$ exists.

$\{\lvert u_1\rvert, \lvert u_2\rvert, \lvert u_3\rvert\}$	ℓ^2
$\{1,0,1\}$	2
$\{2,0,2\}$	8
$\{2,1,1\}$	6
$\{3,0,3\}$	18
$\{3,1,2\}$	14
$\{4,0,4\}$	32
$\{4,1,3\}$	26
$\{4,2,2\}$	24

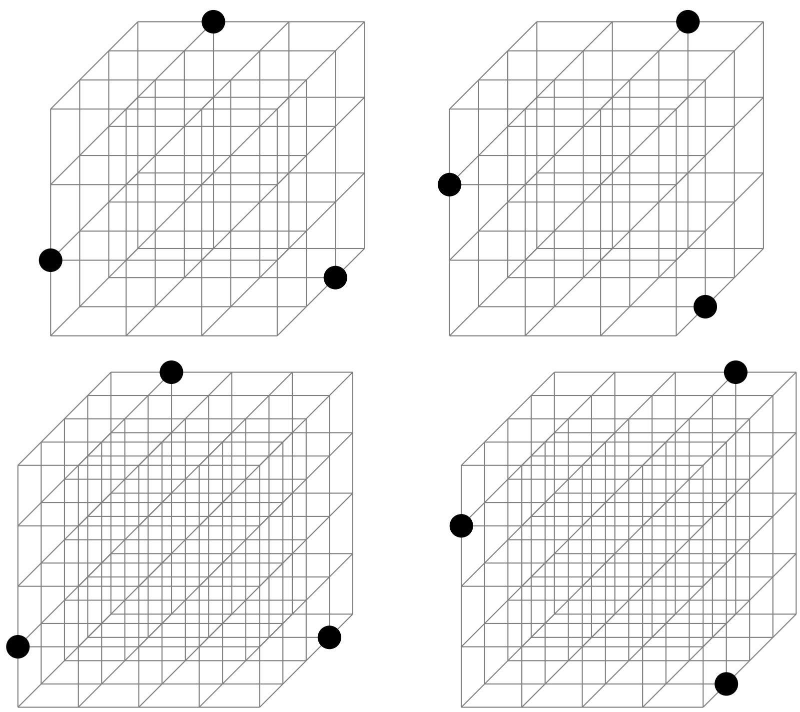
Now we can count how many of each of these triangles there are.

$\{1,0,1\}$: Three vectors of this form correspond to corners on a unit cube, and as seen in the ciphering round there are 8 such triangles in each unit cube. The scaled versions of $\{1,0,1\}$, namely $\{2,0,2\}$, $\{3,0,3\}$, and $\{4,0,4\}$, correspond to the corners of a $2\times2\times2$ cube, a $3\times3\times3$ cube, and $4\times4\times4$ cube respectively. In the $4\times4\times4$ cube there are 64 copies of a unit cube, 27 copies of a $2\times2\times2$ cube, and 8 copies of a $3\times3\times3$ cube so collectively these contribute $(64+27+8+1)8 = 800$ triangles.

$\{2,1,1\}$: Three vectors of this form correspond to choosing the midpoints of edges of a $2\times2\times2$ cube as shown in example below. Each is positioned symmetrically with respect to a vertex (a specific knight's move away), so there are 8 for each $2\times2\times2$ subcube. There are also 8 from the scaled version $\{4,2,2\}$ in the $4\times4\times4$ cube. This adds a total of $(27+1)8 = 224$ triangles



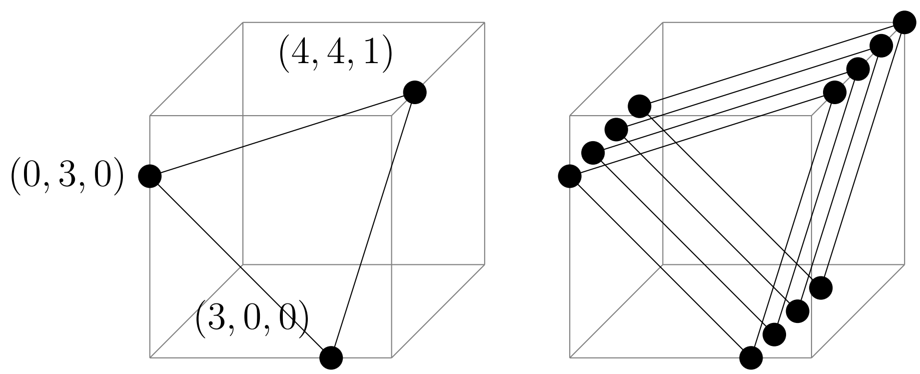
$\{3,1,2\}$. This corresponds to two different equilateral triangles symmetric with respect to a vertex as pictured below (for a total of 16 in each $3\times3\times3$ subcube). The case $\{4,1,3\}$ is essentially identical (also pictured below). These add a total of $(8+1)16 = 144$ new triangles.



Case 2: Using both $\{3,0,3\}$ and $\{4,1,1\}$. It is not possible to use two $\{3,0,3\}$ coordinate magnitude vectors, but it *is* possible to use two $\{4,1,1\}$ coordinate magnitude vectors. One working collection is

$$\vec{u}=(4,1,1),\quad \vec{v}=(-1,-4,-1),\quad \vec{u}=(-3,3,0),$$

which corresponds to the type of triangle pictured below. Observe that you can "push" this triangle into the cube 4 times to get 4 new triangles of the same orientation given a starting choice of the $\{3,0,3\}$ coordinate magnitude vector on a face of the cube.



For each of the 6 faces of the $4 \times 4 \times 4$ cube, there are 4 ways of picking the $\{3, 0, 3\}$ coordinate magnitude vector, leading to a total of $6 \cdot 4 \cdot 4 = 96$ additional triangles.

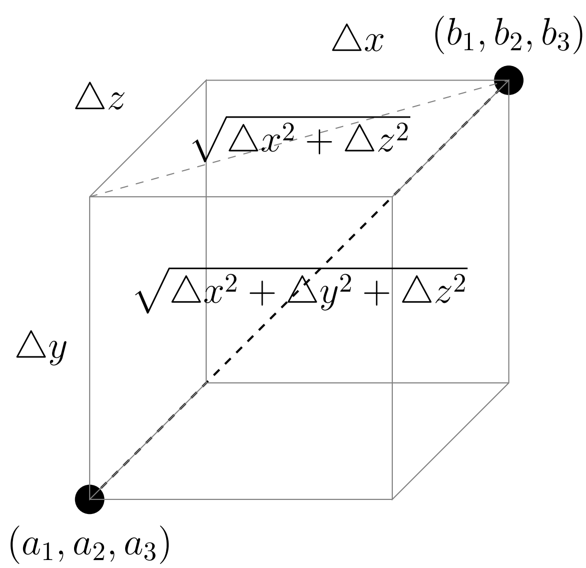
Adding up our counts, we obtain the total number of equilateral triangles as

$$800 + 224 + 144 + 96 = 1264.$$

($*$) You can see the three dimensional distance formula by either noting that the equation for a sphere of radius r centered at (a_1, a_2, a_3) is

$$(x - a_1)^2 + (y - a_2)^2 + (z - a_3)^2 = r^2,$$

or by drawing right triangles and applying the Pythagorean theorem twice as below (with $\triangle x = b_1 - a_1$, $\triangle y = b_2 - a_2$, $\triangle z = b_3 - a_3$).



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