1785

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 20, 2018

WITH SOLUTIONS

Problem 1. If $a > 2$ and $(a-2)^a + (a-1)^a + (a+1)^a + (a+2)^a = 2018$, what is a?

(A) 3 (B)$^\heartsuit$ 4 (C) 5 (D) 6 (E) 7

Solution. If $a = 3$ then the biggest term in the sum is $5^3 = 125$, so the total sum is less than $4 \times 125 = 500$. If $a \geq 5$, then the largest term in the sum is at least $7^5 > 7^4 = 2401$ so the sum is too large. Thus of the answer choices, 4 is the only one possible, and indeed

$$2^4 + 3^4 + 5^4 + 6^4 = 2018.$$

Problem 2. A certain recipe calls for butter, eggs, sugar, vanilla, and flour. After the other ingredients are well mixed, the recipe says: "Add $\frac{1}{3}$ of the flour and mix well. Then add $\frac{1}{2}$ the remaining flour and mix well. Then add the rest of the flour, mix well, and bake."

If "the rest of the flour" is 1 cup, how much flour is in the recipe?

(A) $1\frac{1}{2}$ cups (B) $1\frac{2}{3}$ cups (C) $1\frac{5}{6}$ cups (D) 2 cups (E)$^\heartsuit$ 3 cups

Solution. If the recipe calls for n cups of flour, then the first addition is $\dfrac{1}{3}n$ cups, so there are $\dfrac{2}{3}n$ cups flour remaining. The second addition is $\dfrac{1}{2}\left(\dfrac{2}{3}n\right) = \dfrac{1}{3}n$ cups, so now there are $\dfrac{1}{3}n$ cups remaining. "The rest of the flour" is then $\dfrac{1}{3}n = 1$ cup, so $n = 3$ cups.

Problem 3. Each of the following rows contains two functions. For which row(s) are the graphs of the two functions identical?

I. $y = \log\left((x+5)(x^2-16)\right)$, $y = \log(x+5) + \log\left(x^2-16\right)$

II. $y = \log\left((x+5)(x^2-16)\right), \quad y = \log(x+5) + \log(x-4) + \log(x+4)$

III. $y = \log\left((x+3)(x^2-16)\right), \quad y = \log(x+3) + \log\left(x^2-16\right)$

IV. $y = \log\left((x+3)(x^2-16)\right), \quad y = \log(x+3) + \log(x-4) + \log(x+4)$

(A)$^\heartsuit$ **I** (B) **II** (C) **III** (D) **IV** (E) The graphs are identical in each row.

Solution. Rules of logarithms state that in each line the two functions are the same *wherever they are both defined.* Thus this is really a question of domain.

Recalling that log is only defined on positive numbers, we see that problems may occur when some of the inner functions above take on negative values. Indeed for **II** the left hand function is defined at $x = -4.5$ while the right hand side is not. For **III** and **IV** the left hand functions are defined at $x = -3.5$ while the right hand functions are not.

For **I** the left hand function is defined whenever $(x+5)\left(x^2-16\right) > 0$ which is when $-5 < x < -4$ or $4 < x$. The right hand side is defined when both $x > -5$ (from $\log(x+5)$) and $x < -4$ or $4 < x$ (from $\log\left(x^2-16\right)$). This is exactly the same as $-5 < x < -4$ or $4 < x$.

Problem 4. 100 students took a test, and their average score was 75 (out of a possible 100). If n is the number of students who scored ≥ 90 what is the largest n can be?

(A) 50 (B) 75 (C) 80 (D)$^\heartsuit$ 83 (E) 84

Solution. The hundred students scored a combined 7500 points. Since $\dfrac{7500}{90} = 83.\overline{3}$, there could be as many as 83 scores of 90. That only leaves 30 points unaccounted for, so the remaining 17 people had an average under 2.

Problem 5. Suppose $x_1, x_2, \ldots, x_n$ is a finite data set with

$$\text{minimum} < \text{mean} < \text{median} < \text{mode} < \text{maximum}.$$

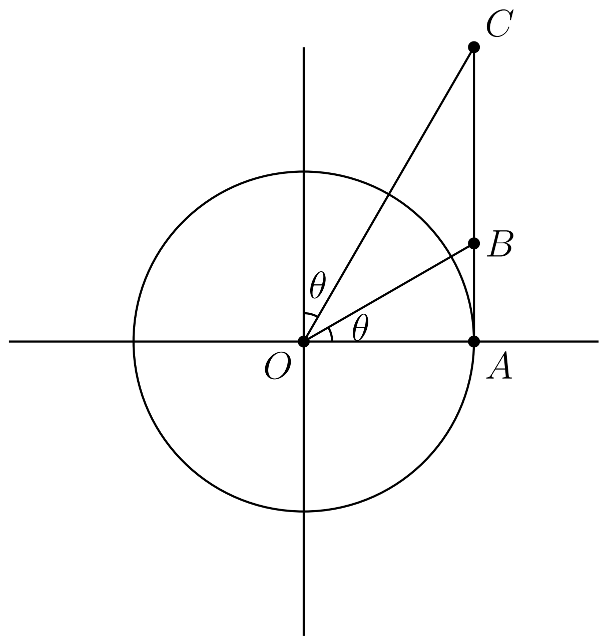
What is the smallest n can be?

We use the following definitions: When the data are listed in increasing order, the median is the middle number in that list, or the average of the two middle numbers. The mode is the unique data point that occurs most frequently.

(A) 4 (B) 5 (C)$^\heartsuit$ 6 (D) 7 (E) there is no such data set

Solution. Any such data set having a mode must contain at least the distinct numbers m = minimum, M = maximum, and (twice) a = mode. If $n = 4$ or $n = 5$, a is also the median. With $n = 6$ data points, the result is possible; e.g. the data set 0, 1, 2, 3, 3, 4 has min = 0, mean = $2.1\overline{6}$, median = 2.5, mode = 3, and max = 4.

Problem 6. In the diagram shown $0 < \theta < \dfrac{\pi}{4}$ and the line AC is tangent to the circle at A. Express $\dfrac{OC}{OB}$ as a function of θ.

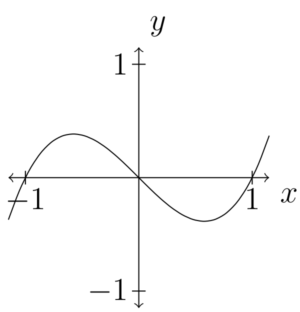


(A) $\dfrac{\sec(\theta)}{\sqrt{2}}$ (B) $\dfrac{\csc(\theta)}{\sqrt{2}}$ (C) $\tan(\theta)$ (D)$^{\heartsuit}$ $\cot(\theta)$ (E) $\dfrac{\pi}{4\theta}$

Solution. Notice that $\angle C = \theta$ (because of transverse angles). Therefore $\triangle OBA$ is similar to $\triangle COA$. Thus,

$$\frac{OC}{OB} = \frac{OA}{AB} = \cot(\theta).$$

Problem 7. Teacher: Here is the graph of $f(x)$.



Teacher: What does the graph of $f\left(x^2\right)$ look like?
Me:

(A) Top left (B) Top right (C) Bottom left (D)$^{\heartsuit}$ Bottom right

(There are only 4 answer choices for this problem.)

Solution. Notice that if $0 < x < 1$, then $0 < x^2 < 1$, so $-1 < f\left(x^2\right) < 0$. This eliminates the left options. For x near 0, $f(x)$ looks approximately like $-x$, so $f\left(x^2\right)$ will look like $-x^2$, which eliminates the top right.

Problem 8. Suppose x_n is a sequence of integers which satisfy the usual Fibonacci recurrence:

$$x_{n+1} = x_n + x_{n-1}.$$

What is the smallest possible value of x_1 if x_1 is positive and $x_1 = x_{10}$?

(A) 7 (B) 13 (C) 14 (D)$^{\heartsuit}$ 17 (E) 34

Solution. If $x_1 = y$ and $x_2 = x$, then the sequence is

$$y,\ x,\ x+y,\ 2x+y,\ 3x+2y,\ 5x+3y,\ 8x+5y,\ 13x+8y,\ 21x+13y,\ 34x+21y,$$

so we want $34x + 21y = y$; i.e. $\frac{x}{y} = -\frac{10}{17}$. Thus 17 divides y, and $y = 17$, $x = -10$ produces the valid solution

$$17,\ -10,\ 7,\ -3,\ 4,\ 1,\ 5,\ 6,\ 11,\ 17.$$

Problem 9. A chocolate bar is a rectangle made up of individual square pieces. You can break a bar into two smaller bars by separating along any row or column that joins pieces together. For this problem, a break can *only affect one bar at a time*; for example you may not stack different bars on top of each other to perform simultaneous breaks.

If we start with a 3×4 rectangle, what is the minimum number of breaks required to separate all 12 individual squares?

(A) 8 (B) 9 (C) 10 (D)$^{\heartsuit}$ 11 (E) 12

Solution. For each new break performed, the total number of sub-bars increases by 1. Therefore, to go from the whole bar to 12 individual 1-piece bars, you must use 11 breaks.

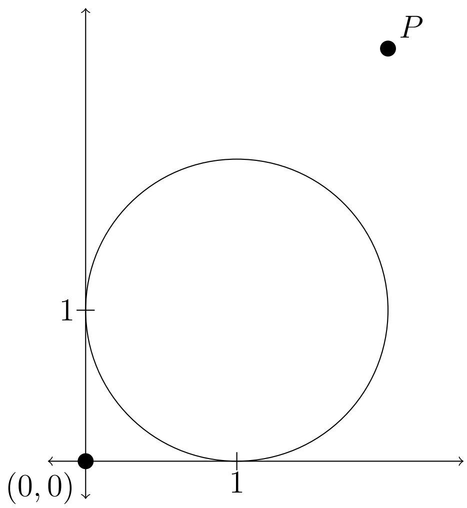
Problem 10. For this problem, we want to break a chocolate bar into individual pieces, but we *can* stack different bars on top of each other to break multiple bars simultaneously. With that change to the rules, what is the minimum number of breaks needed to completely separate a 3×4 bar?

(A) 3 (B)$^{\heartsuit}$ 4 (C) 5 (D) 6 (E) 7

Solution. For a general $m \times n$ bar, where we (without loss of generality) break up the columns, at least one sub-bar will have $\geq \frac{n}{2}$ columns. Therefore, when we perform b total column breaks, there must be a piece with at least $\frac{n}{2^b}$ columns, and thus you need at least $\lceil \log_2(n) \rceil$ breaks to separate all the columns. Similarly, you need at least $\lceil \log_2(m) \rceil$ breaks to separate all the rows, so the sum of these ceilings is a lower bound on the problem. For the 3×4 bar, that bound is $2 + 2 = 4$.

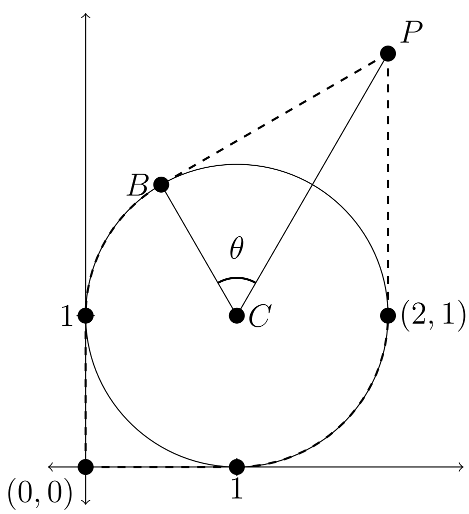
Indeed 4 is attainable: first do a break down the middle to get two 3×2 bars, then stack them and break them in the middle to get four 3×1 bars, and then stack those four on each other and perform two breaks through each simultaneously.

Problem 11. The circle shown has radius 1 and center $(1, 1)$. What is the length of the shortest path from $(0, 0)$ to $P = (2, 1 + \sqrt{3})$ that does not go inside the circle. The path may touch the circle.



(A) $1 + \frac{\pi}{2} + \sqrt{3}$ (B)♡ $1 + \frac{\pi}{3} + \sqrt{3}$ (C) $1 + \frac{\pi}{4} + \sqrt{3}$
(D) $1 + \frac{\pi}{6} + \sqrt{3}$ (E) $1 + \frac{\pi}{2} + \sqrt{5 - 2\sqrt{3}}$

Solution. There are two reasonable candidates for shortest path:



1. $(0,0) \to (1,0) \to (2,1) \to P$
2. $(0,0) \to (0,1) \to B \to P$ where B is the other point on the circle whose tangent line goes through P.

Notice that the first and last steps in each path have the same length. Thus the path through B is the shortest because it travels around less of the circle. To find this distance, notice that $\triangle CPB$ is a right triangle with hypotenuse $CP = 2$ and $CB = 1$, so $\theta = \frac{\pi}{3}$. By symmetry, the angle at C in the triangle formed by $(2,1)$, C, and P is also θ, so the angle through which the path turns is $\pi - 2\theta = \frac{\pi}{3}$, and the distance traveled around the circle is also $\frac{\pi}{3}$. Adding this to the distance from $(0,0) \to (0,1)$

and the distance from $B \to P$ (which is conveniently the same as the distance from $(2,1) \to P$) gives a total distance of $1 + \frac{\pi}{3} + \sqrt{3}$.

Problem 12. How many ordered pairs (x, y) of distinct positive integers satisfy

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{2018}?$$

(A) 0 (B) 4 (C)$^{\heartsuit}$ 8 (D) 16 (E) 20

Solution. We first count the number of such pairs ignoring the condition that $x \neq y$. It is clear that if x, y satisfy the given equation, then $x, y > 2018$. Now multiplying through by $2018xy$ and rearranging shows that we are looking for the number of pairs of positive integers $x, y > 2018$ satisfying

$$(x - 2018)(y - 2018) = 2018^2.$$

But this is just the number of positive integer divisors of 2018^2. Factoring 2018^2 into primes yields $2^2 \cdot 1009^2$, and this has 9 positive divisors: $2^a \cdot 1009^b$ for $0 \leq a, b \leq 2$. So there are 9 ordered pairs of x, y satisfying our equation. Removing the pair $x = y = 2 \cdot 2018$ leaves 8 pairs with x, y distinct.

Problem 13. Suppose you're going to choose an integer from 1 to 100, inclusive, and that for each $k = 1, 2, \ldots, 100$, you are k times as likely to choose the number k as you are to choose the number 1. What is the expected value of your choice?

(A) 50 (B) $66.\overline{6}$ (C)$^{\heartsuit}$ 67 (D) 75 (E) 80

Solution. For $1 \leq k \leq 100$, let p_k be the probability that you pick k. Then $p_k = kp_1$, and

$$1 = \sum_{1}^{100} p_k = \sum_{1}^{100} kp_1 = p_1 \frac{100 \cdot 101}{2},$$

so $p_1 = \frac{2}{100 \cdot 101}$. The expected value is then

$$\sum_{1}^{100} kp_k = \sum_{1}^{100} k^2 p_1 = \frac{2}{100 \cdot 101} \cdot \frac{100 \cdot 101 \cdot 201}{6} = \frac{201}{3} = 67.$$

Problem 14. Call a composite number *obviously composite* if it is divisible by 2, 3, or 5. How many composite numbers are there in the interval $[2, 1000]$ that are not obviously composite? You may find helpful that the interval $[2, 1000]$ contains 168 prime numbers.

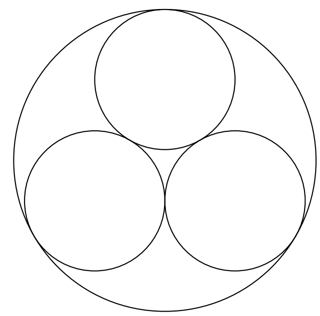
(A) 97 (B)$^\heartsuit$ 100 (C) 222 (D) 266 (E) 267

Solution. Applying the principle of inclusion-exclusion, we find that the number of integers in $[2, 1000]$ that are divisible by at least one of $2, 3$, or 5 is

$$[1000/2] + [1000/3] + [1000/5] - [1000/6] - [1000/10] - [1000/15] + [1000/30] = \\ 500 + 333 + 200 - 166 - 100 - 66 + 33 = 734.$$

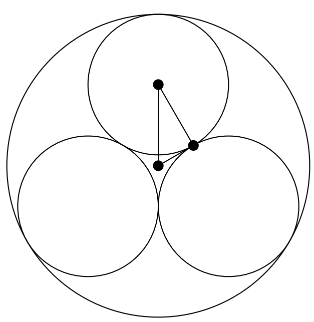
Of these 734 numbers, 731 of them are composite (we must remove the three primes $2, 3$, and 5 from the count). Since there are a total of $999 - 168 = 831$ composites in the interval $[2, 1000]$, the number of nonobvious composites there is $831 - 731 = 100$.

Problem 15. A circle of radius 1 has three congruent mutually tangent circles inscribed in it as shown. What is the radius r of the inscribed circles?



(A) $\frac{1}{3}$ (B) $2 - \sqrt{3}$ (C) $\sqrt{2} - 1$ (D)$^\heartsuit$ $2\sqrt{3} - 3$ (E) $3 - 2\sqrt{2}$

Solution. Notice the centers of the circles form an equilateral triangle. Therefore the triangle drawn below is a 30-60-90 right triangle.



If r is the radius of an inscribed circle, then by the properties of 30-60-90 right triangles, the distance from the center of an inscribed circle to the center of the radius 1 circle is $\frac{2r}{\sqrt{3}}$. Therefore we have the equation

$$r + \frac{2r}{\sqrt{3}} = 1.$$

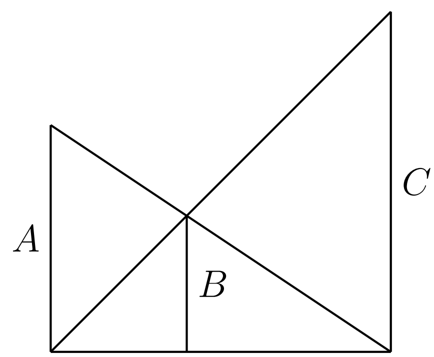
Rearranging this equation gives

$$r = \frac{\sqrt{3}}{2 + \sqrt{3}} = \frac{\sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} = 2\sqrt{3} - 3.$$

Alternatively, Descartes' Circle Theorem says (if R is the radius of the outer circle and r_1, r_2, r_3 the radii of the inscribed circles)

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{R}\right)^2 = 2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{R^2}\right).$$

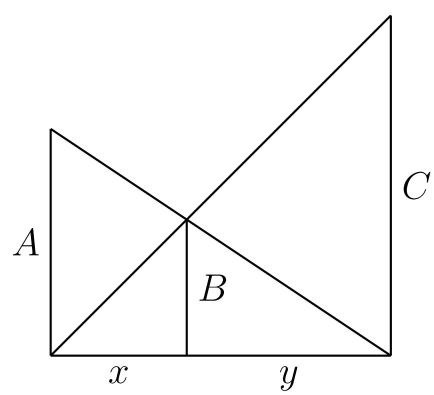
Problem 16. If A, B, and C are the side lengths shown in the diagram, which of the following is true?



(A) $A+C=4B$ (B) $AC=4B^2$ (C) $A^2+C^2=4B^2$

(D)$^\heartsuit$ $\frac{1}{A}+\frac{1}{C}=\frac{1}{B}$ (E) $\frac{AB}{2}+\frac{BC}{2}=AC$

Solution. First we write the auxiliary lengths x and y as shown below.



By similar triangles:

$$\frac{B}{A}=\frac{y}{x+y},$$

and

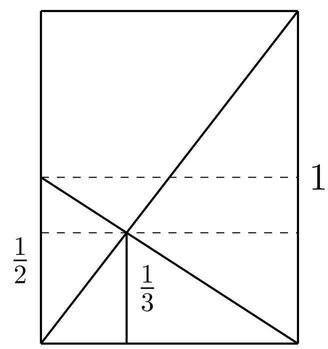
$$\frac{B}{C}=\frac{x}{x+y}.$$

Adding these equations, we get

$$B\left(\frac{1}{A}+\frac{1}{C}\right)=\frac{y}{x+y}+\frac{x}{x+y}=\frac{x+y}{x+y}=1,$$

and rearranging gives the answer.

Remark. Notice that if we fix A and let $C \to \infty$, then $B \to A$, which eliminates all choices but the answer. As an interesting application, notice that $A = \dfrac{1}{2}$ and $C = 1$ implies $B = \dfrac{1}{3}$. It's easy to see how to fold a piece of paper in half; this shows how to fold a piece of paper into thirds:



Problem 17. Call a prime *deletable* if it remains prime upon deletion of any proper subset of its (base 10) digits. How many deletable primes exist? *Note: A proper subset means a subset that is not the entire set.*

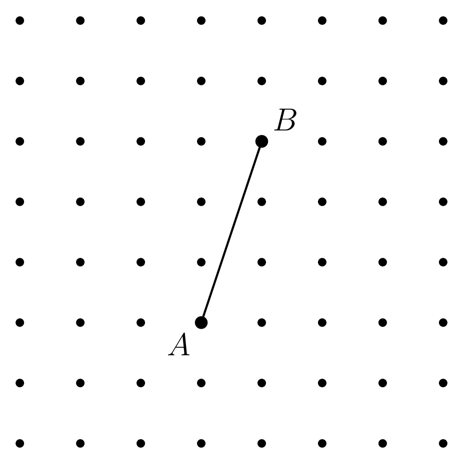
(A) 7 (B)$^\heartsuit$ 8 (C) 9 (D) 10 (E) 12

Solution. We make several observations:

- We can always remove digits so that there is only a single digit left. Therefore all the digits themselves better be primes, i.e. the digits come from the set $\{2, 3, 5, 7\}$.
- We cannot abide repeated digits as deleting down to two repeated digits yields a number divisible by 11.
- If a deletable prime contains a 2 or 5, then that 2 or 5 must appear at the beginning of the prime (otherwise we could delete down to a two digit number ending in 2 or 5). In particular a deletable prime cannot have both a 2 and a 5.
- Consideration modulo 3 tells us a deletable prime cannot have a 7 with a 2 or a 5 (that is any two digit number with those digits is divisible by 3).
- Therefore if a deletable prime has a 7, it can have at most 2 digits. On the other hand, if it does not contain a 7, then it can also have at most two digits because it cannot contain both 2 and 5.

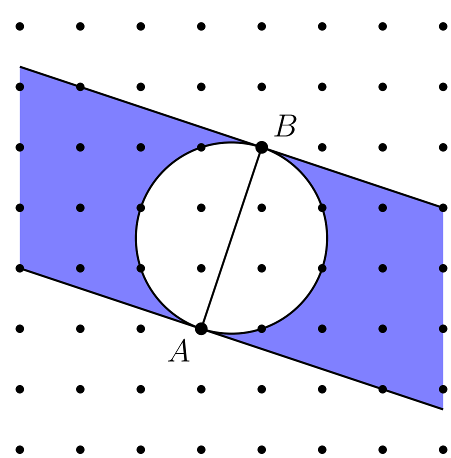
- We have reduced to checking all two digit numbers which fit our constraints: If the deletable prime has a 7, then it cannot have a 2 or 5, so we have two possibilities: 37 and 73, both valid. If it does not contain a 7, then it must start with 2 or 5 and end in 3, giving 23 and 53 both of which are also valid. Adding these four solutions to the single digit primes which are automatically valid gives 8 deletable primes.

Problem 18. In the 8 by 8 grid below two points A and B are marked. If point C is one of the other points in the grid, how many choices of C make the triangle ABC acute (i.e. every angle less than 90 degrees)?



(A) 8 (B)♡ 14 (C) 18 (D) 20 (E) 30

Solution. If ABC is acute, then each angle needs to be less than 90°. In order for both $\angle A$ and $\angle B$ to be less than 90°, C has to lie in the strip between A and B (i.e. between the lines perpendicular to segment AB). In order for $\angle C$ to be acute, it needs to lie outside the circle through A and B with center directly between A and B (recall any point on that circle will form a right triangle with A and B). The remaining region shaded below contains 14 points.



Problem 19. What is the largest prime p such that p^2 divides the binomial coefficient $\binom{100}{50}$?

(A)$^\heartsuit$ 3 (B) 7 (C) 19 (D) 31 (E) 47

Solution. For any prime p the largest power of p that divides $n!$ is

$$\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots . \qquad (*)$$

Thus if $p^2 > 100$ then the power of p which divides $\binom{100}{50} = \frac{100!}{50!\,50!}$ is

$$\left\lfloor \frac{100}{p} \right\rfloor - 2\left\lfloor \frac{50}{p} \right\rfloor .$$

For any x note that $\frac{x}{p} - 1 < \left\lfloor \frac{x}{p} \right\rfloor \le \frac{x}{p}$. Hence

$$\left\lfloor \frac{100}{p} \right\rfloor - 2\left\lfloor \frac{50}{p} \right\rfloor < \frac{100}{p} - 2\left(\frac{50}{p} - 1\right) = 2.$$

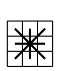
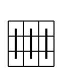
Therefore if p^2 divides $\binom{100}{50}$, then $p \le 10$. Using $(*)$ we can explicitly check the primes 7, 5, and 3 in order, finding that 7 and 5 do not divide $\binom{100}{50}$ while in fact not just 3^2 but rather 3^4 does divide it.

Problem 20. We define a *magic square* to be a collection of nine entries in a 3×3 grid such that the three numbers in each row, column, and diagonal add up to be the same value. If you have a magic square and you know the first two entries in the middle row are 3 and 21 as shown, what is the third entry in that row?

3	21	?

(A) 3 (B) 8 (C) 23 (D)$^\heartsuit$ 39 (E) there is more than one possibility

Solution. Let S be the *magic sum*, the value to which all the entries of each row, column, and diagonal add up. Let c be the center value in the magic square. Then

$4S =$  $=$  $+ 3c = 3S + 3c,$

so $S = 3c$. In our case $c = 21$, so $S = 63$. Thus

$$? = 63 - 21 - 3 = 39.$$

Technically we have not yet shown the existence of a solution. There are many solutions, but here is one with some symmetry:

30	21	12
3	21	39
30	21	12

Problem 21. A polynomial $f(x)$ of degree 4, with real number coefficients, has the property that $f(n)$ is an integer whenever n is an integer. Write

$$f(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0.$$

If $0 \le a_3 \le 1$, then how many possible values are there for a_3?

(A) 1 (B) 2 (C) 6 (D) 7 (E)$^{\heartsuit}$ 13

Solution. Call a polynomial *integer-valued* if it takes integer values at all integer inputs. For each nonnegative integer k, define the polynomial $\binom{x}{k}$ as follows:

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}.$$

When $k = 0$, the product in the numerator is empty, and we understand $\binom{x}{0}$ to be equal to 1. It is clear that $\binom{x}{k}$ always has rational number coefficients. Moreover, $\binom{x}{k}$ is integer-valued. To see why the last statement is true, note first that if n is a nonnegative integer, then $\binom{x}{k}\big|_{x=n} = \binom{n}{k}$, where the right-hand side is the usual binomial coefficient. But binomial coefficients are obviously integers (since they count something). It follows that for all nonnegative integers n, the integer $n(n-1)\cdots(n-k+1)$ is always a multiple of $k!$. But the value of $n(n-1)\cdots(n-k+1)$ modulo $k!$ depends only on n modulo $k!$ — so if it is 0 for all nonnegative integers n, it is in fact 0 for *all* integers n. Thus, $\binom{x}{k}\big|_{x=n}$ is an integer for all integers n. As a consequence, for any finite sequence of integers $a_0, a_1, \ldots, a_k$, the polynomial

$$a_0\binom{x}{0} + a_1\binom{x}{1} + \cdots + a_k\binom{x}{k}$$

is an integer-valued polynomial.

Claim: Every integer valued polynomial has the form just described.

Indeed, let $f(x)$ be an integer valued polynomial of degree d. We can choose constants $c_0, c_1, \ldots, c_d$ with

$$f(x) = c_0 + c_1\binom{x}{1} + \cdots + c_d\binom{x}{d}.$$

(First choose c_d to make the coefficients of x^d match, then c_{d-1} to match up the coefficients of x^{d-1}, etc.) To prove the claim, it is enough to show that all of $c_0, \ldots, c_d$ are integers. Plugging in $x = 0$ shows c_0 is an integer, since $c_0 = f(0)$ and f is integer-valued. Now plugging in $x = 1$ shows that $c_1 + c_0$ is an integer; since c_0 is already known to be an integer, we find that c_1 is also an integer. Continuing in this way, plugging in $x = 2, \ldots, d$, we find that all of $c_0, c_1, \ldots, c_d$ are integers, as desired.

Getting back to the original problem, we see that f being integer-valued of degree 4 amounts to saying

$$f(x) = c_0 + c_1\binom{x}{1} + c_2\binom{x}{2} + c_3\binom{x}{3} + c_4\binom{x}{4},$$

where the c_i are integers and $c_4 \neq 0$. Expanding, the coefficient of x^3 in $f(x)$ has the form

$$\frac{-3c_4 + 2c_3}{12}.$$

If this is between 0 and 1 (inclusive), it is obviously one of the numbers $a/12$ for $a = 0, \ldots, 12$. Moreover, all of these are possible for some choice of the c_i. (To get $a = 0$, take $c_4 = 2, c_3 = 3$. To get any other a, take $c_4 = c_3 = -a$.) So there are 13 possible values for the coefficient of x^3.

Problem 22. If $x + \dfrac{1}{x} = 3$, what is $x^{12} + \dfrac{1}{x^{12}}$?

(A)$^\heartsuit$ 103682 (B) 103729 (C) 103822 (D) 103823 (E) 104974

Solution. It is possible to solve this problem by first solving for x in the first equation and plugging the result into the second equation. It is also possible to solve the problem by raising the first equation to the 12th power and reducing the problem to computing $x^n + \dfrac{1}{x^n}$ for even values of n less than 12. However, both of these methods are extremely painful computationally. Below we will outline two computation methods which are not nearly so awful.

First for convenience we will make the definition

$$X_n := x^n + x^{-n}.$$

Note with this definition $X_0 = 2$, and the original question can be written as: If $X_1 = 3$, what is X_{12}?

Notice that

$$X_n X_m = x^{n+m} + x^{n-m} + x^{m-n} + x^{-n-m} = X_{n+m} + X_{n-m}. \tag{1}$$

In particular if $n = m$, this tells us

$$X_n^2 = X_{2n} + 2. \tag{2}$$

Together with (1), this suggests two possible paths of computation:

$$X_1 \to X_2 \to X_3 \to X_6 \to X_{12} \quad \text{or} \quad X_1 \to X_2 \to X_4 \to X_8 \to X_{12}.$$

Using (2) we can compute $X_2 = X_1^2 - 2 = 3^2 - 2 = 7$ (as you may have done on the ciphering).

Now for the first path, using (1) with $n = 2$, $m = 1$ compute

$$X_3 = X_2 X_1 - X_1 = 7 \times 3 - 3 = 18.$$

Using (2) again we compute

$$X_6 = X_3^2 - 2 = 18^2 - 2 = 324 - 2 = 322.$$

Finally using (2) one last time we compute

$$X_{12} = X_6^2 - 2 = 322^2 - 2 = 103682.$$

Alternatively, (for the second path) by (1) with $n = 8$ and $m = 4$, we can write

$$X_{12} = X_8 X_4 - X_4 = (X_8 - 1)\, X_4,$$

so we can reduce the question to computing X_4 and X_8, which is possible directly with (2):

$$X_4 = X_2^2 - 2 = 7^2 - 2 = 47,$$
$$X_8 = X_4^2 - 2 = 47^2 - 2 = 2207,$$

and thus we again get

$$X_{12} = (X_8 - 1)\, X_4 = 2206 \times 47 = 103682.$$

Problem 23. The UGA MathClub is proud to announce it has its own cryptarithmetic puzzle:

$$\begin{array}{r} U\,G\,A \\ +\,H\,S\,M\,T \\ \hline M\,A\,T\,H \end{array}$$

A cryptarithmetic puzzle is an arithmetic problem, as above, where each letter represents a single digit (0 - 9). Each occurrence of the same letter must represent the same digit; different letters represent different digits. Also, there are no leading 0's.

If, in the puzzle above, $A = 7$, what is M?

(A) 1 or 2 (B)$^\heartsuit$ 3 or 4 (C) 5 or 6 (D) 7 or 8 (E) 9 or 0

Solution. Some observations in order:

- Since $H \neq M$, we must have $M = H + 1$ and $U + S$ must result in a carry.
- Since $A = 7$, we must have $U + S = 16$ with a carry from the second column, or $U + S = 17$ without a carry. The first of these would require $\{U, S\} = \{7, 9\}$ or $\{U, S\} = \{8, 8\}$, neither of which is possible. So $U + S = 17$ and $\{U, S\} = \{8, 9\}$.

  So far we have

$$\begin{array}{r} 8\,G\,7 \\ +\ H\,9\,M\,T \\ \hline M\,7\,T\,H \end{array}$$

  (where the 8 and 9 are possibly reversed).
- Next, looking at the ones column, notice that $T \leq 2$ implies $H \geq 7$, but 7, 8, and 9 are already used.
- Also notice $T = 3$ implies $H = 0$, but this is impossible since $HSMT$ would begin with 0.
- So T can only be 4, 5, or 6.
- So H can only be 1, 2, or 3.
- So M can only be 2, 3, or 4.

Let's check $M = 2$:

$$\begin{array}{r} 8G7 \\ +\ 1924 \\ \hline 2741 \end{array}$$

This would require $G = 1$, which is not possible. Thus M must be 3 or 4, and in fact both are possible:

$$\begin{array}{r} 817 \\ +\ 2935 \\ \hline 3752 \end{array}$$

$$\begin{array}{r} 817 \\ +\ 3946 \\ \hline 4763 \end{array}$$

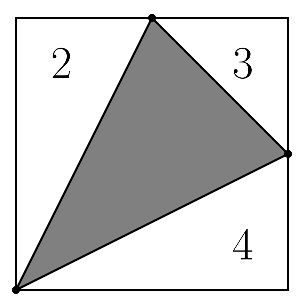
Problem 24. Suppose
$$\sqrt[3]{\frac{a}{b}} = \sqrt[3]{3} + \left(3\sqrt[3]{2} - 3\right)^{\frac{2}{3}},$$
where $\frac{a}{b}$ is a rational number in lowest terms. What is $a + b$?

(A)$^\heartsuit$ 13 (B) 29 (C) 31 (D) 42 (E) 52

Solution. Let $x = \sqrt[3]{2}$. Then
$$(2 - x)^3 = 8 - 12x + 6x^2 - x^3 = 8 - 12x + 6x^2 - 2 = 6(x - 1)^2.$$
Thus
$$\frac{2 - x}{\sqrt[3]{6}} = (x - 1)^{\frac{2}{3}}.$$

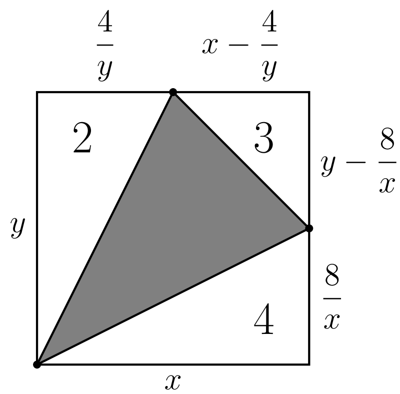
Writing 2 as $\sqrt[3]{8}$ and $x = \sqrt[3]{2}$ gives
$$\sqrt[3]{\frac{4}{3}} - \sqrt[3]{\frac{1}{3}} = \left(\sqrt[3]{2} - 1\right)^{\frac{2}{3}}.$$
Multiplying through by $\sqrt[3]{9}$, we get
$$\sqrt[3]{12} - \sqrt[3]{3} = \left(3\sqrt[3]{2} - 3\right)^{\frac{2}{3}},$$
so $\frac{a}{b} = \frac{12}{1}$ and the answer is $12 + 1 = 13$.

Problem 25. In the rectangle shown at right the three border triangles have areas 2, 3, and 4 as shown (though the figure is certainly not drawn to scale). What is the area of the shaded central triangle?



(A) 5 (B) 6 (C)$^\heartsuit$ 7 (D) 8 (E) 9

Solution. Calling the length x and the height y, we can draw in some edge lengths using the $A = \frac{1}{2}bh$ formula for a triangle:



Now we have a relation from the upper right triangle, namely

$$3 = \frac{1}{2}\left(x - \frac{4}{y}\right)\left(y - \frac{8}{x}\right).$$

Multiplying through by 2 and expanding gives

$$6 = xy - 4 - 8 + \frac{32}{xy}.$$

Multiplying through by xy yields a quadratic equation in xy:

$$0 = (xy)^2 - 18xy + 32 = (xy - 16)(xy - 2).$$

Therefore $xy = 2$ or $xy = 16$. However, we can recognize xy as the area of the rectangle, so $xy = 2$ is an extraneous solution and the true area of the rectangle is 16. Thus the area of the shaded triangle is $16 - 2 - 3 - 4 = 7$.

Authors. Written by Mo Hendon, Paul Pollack, and Peter Woolfitt. Problems #9 and #10 were suggested by Michael Klipper. Problem #14 is based on observations of Alf van der Poorten and John H. Conway; see `http://www.austms.org.au/Gazette/2006/Nov06/vanderpoorten.pdf`