I

AMERICAN MATHEMATICAL SOCIETY

Lecture Notes Prepared in Connection With the

Summer Institute on

Algebraic Geometry

held at the

Whitney Estate, Woods Hole, Massachusetts

July 6 - July 31, 1964

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REVISED PROGRAM OF THE
SUMMER INSTITUTE IN ALGEBRAIC GEOMETRY

(July 6 - July 31, 1964)

Monday, July 6: REGISTRATION DAY.

I THEORY OF SINGULARITIES.

Tuesday, July 7:

10:00 - 11:00 a.m.	S. Abhyankar. Current status of the resolution problem.
11:30 - 12:30 p.m.	H. Hironaka. Equivalences and deformations of isolated singularities.
4:30 - 5:30 p.m.	O. Zariski. Equisingularity and related questions of classification of singularities.

II. CLASSIFICATION OF SURFACES AND MODULI.

Wednesday, July 8:

10:00 - 11:00 a.m.	K. Kodaira. On the structure of compact complex analytic surfaces.
11:30 - 12:30 p.m.	T. Matsusaka. Deformations and varieties of moduli.
4:30 - 5:30 p.m.	D. Mumford. The boundary points of moduli schemes.

Thursday, July 9:

10:00 - 11:00 a.m.	M. Nagata. Invariants of a group in an affine ring.
11:30 - 12:30 p.m.	M. Rosenlicht. Transformation spaces, quotient spaces, and some classification problem.

Wednesday, July 15:

4:30 - 5:30 p.m	J. Igusa. On the Siegel modular variety.

III. GROTHENDIECK COHOMOLOGY.

Friday, July 10:

10:00 - 11:00 a.m.	M. Artin. Etale cohomology of schemes.

11:30 - 12:30 p.m.	J. L. Verdier. A duality theorem in the etale cohomology of schemes.
4:30 - 5:30 p.m.	J. Tate. Etale cohomology over number fields.

IV. ZETA FUNCTIONS AND ARITHMETIC OF ABELIAN VARIETIES.

Monday, July 13:

10:00 - 11:00 a.m.	J. W. S. Cassels. The arithmetic of elliptic curves and abelian varieties.
11:30 - 12:30 p.m.	B. M. Dwork. (Title not available)
4:30 - 5:30 p.m.	G. Shimura. The Zeta-function of an algebraic variety and automorphic functions.

Tuesday, July 14:

4:30 - 5:30 p.m.	J. P. Serre. L-Series of schemes.

Oscar Zariski, Chairman
Organizing Committee

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CURRENT STATUS OF THE RESOLUTION PROBLEM

by

Shreeram S. Abhyankar

§1. The problem and its history.

The problem can be stated thus:

Resolution Problem. Given a function field K over a pseudogeometric Dedekind domain k does there exist a nonsingular projective model of K over k?

Before giving the history of the problem let us recall the definitions of the terms used above.

DEFINITION. By a Dedekind domain we mean a normal (i.e., integrally closed in its quotient field) noetherian (integral) domain in which every nonzero prime ideal is maximal; note that then any field is a Dedekind domain. A ring (commutative with identity) k is said to be pseudogeometric if k is noetherian and for every prime ideal P in k we have that the integral closure of k/P in any finite algebraic extension of the quotient field of k/P is a finite (k/P)-module. Note that: every field is pseudogeometric; every Dedekind domain of characteristic zero is pseudogeometric; a Dedekind domain k is pseudogeometric if and only if the integral closure of k in any finite algebraic extension of the quotient field of k is a finite k-module.

By an affine ring over a domain k we mean an overdomain of k which is a finitely generated ring extension of k. A local ring (i.e., a noetherian ring with a unique maximal ideal) R is said to be a spot over a domain k if R is the quotient ring A_P of an affine ring A over k with respect to a prime ideal P in A. The significance of the notion of pseudogeometric is the theorem of Nagata to the effect that if k is a pseudogeometric domain then every affine ring over k is pseudogeometric and so is every spot over k. By a function field over a domain k we mean a field K which is a spot over k, i.e., K is the quotient field of an affine ring over k. Given a function field K over a domain k, by a projective model of K over k we mean a nonempty set V of local domains with quotient field K such that there exists a finite number of nonzero elements $x_0, \ldots, x_m$ in K such that $V = \bigcup_{i=0}^{m} V_i$ where V_i is the set of all quotient rings of $k[x_0/x_i, \ldots, x_m/x_i]$ with respect to the various prime ideals in $k[x_0/x_i, \ldots, x_m/x_i]$; V is said to be nonsingular if every element in V is regular.

<u>History</u>. Let K be a function field over a pseudogeometric Dedekind domain k, let n' be the transcendence degree of K over the quotient field of k, and let n be the absolute dimension of K over k, i.e., n = n' if k is a field, and n = 1 + n' if k is not a field. The Resolution Problem has been settled affirmatively in the following cases: For n = 1 the solution is classical. For n = 2 and k = the field of complex numbers, after several possible solutions by the Italians (notably by Alabanese and Levi)

the first rigorous solution was given by Walker in 1935; Walker's solution makes use of the local solution (i.e., solution of the local uniformization problem which is the localized version of the Resolution Problem) for $n = 2$ and $k =$ the field of complex numbers given by Jung in 1908 in the Crelle Journal. When k is a field of characteristic zero, Zariski gave a solution for $n = 2$ in 1939 - 1942 and for $n = 3$ in 1944; in 1940 Zariski also gave a local solution for n arbitrary and $k =$ a field of characteristic zero. For $n = 2$ and $k =$ a perfect field, Abhyankar gave a solution in 1956. Finally, in 1964 Hironaka gave a solution for n arbitrary and $k =$ a field of characteristic zero. All these are publication dates and all the solutions beginning with Walker's appeared in the Annals of Mathematics. For $n = 2$, a rigorous version of Albanese's proof was given by Artin in the spring of 1963 which works when k is an algebraically closed field of characteristic different from 2. In November 1963 I gave a solution when $n = 2$ and k/P is perfect for every maximal ideal P in k; this proof of the Arithmetical Case is being published. In the last few months I have obtained a solution for $n = 3$ and $k =$ an algebraically closed field (of any characteristic); to be on the safe side here I should say that this is a possible solution in the sense that I have proved several pieces and I roughly see how to put them together but as yet I did not have the time to write up these peices systematically and to fit them together. In any case my present investigations have just begun and they would take a year or more to run their full course. So actually I would have been happier to give today's talk after a year or so because then I could have simply said that this is what I can prove and this is what I cannot. Presently I can only say what is cooking. The reason why after a lapse of some eight years I have come back to the resolution problem is twofold. The primary reason was that the fall of 1963 was the first time after 1955 when I got an opportunity to be in Zariski's neighbourhood(not a Zariski neighbourhood); it is a theorem that to resolve singularities it is necessary to be near Zariski; the resolution problem consists of proving the sufficiency of this condition. The secondary reason was that after Hironaka's outstanding work in characteristic zero, I heard a story from several people to the effect: "We have heard that you are planning to take over the work on the problem where Hironaka has left it off". Although I was definitely not the source of this rumour, nevertheless it prompted me to work.

§2. Embedded resolution

Today I shall say nothing about the arithmetical case. Henceforth all varieties will be defined over an algebraically closed ground field of characteristic p which may or may not be zero. Having once stated the problem precisely, henceforth I shall speak quite informally. A common feature of all the above cited proofs is that to prove resolution for dimension n one needs a stronger result for dimension less than n which includes at least the following:

Embedded Resolution. Given a nonsingular projective algebraic variety W of dimension n and given a hypersurface H in W, there exists a composite monoidal transformation $q : W' \longrightarrow W$ such that the total transform $q^{-1}(H)$ of H has only normal crossings.

Concerning the definitions of the terms used above I shall say only this: Given any irreducible subvariety D of an irreducible algebraic variety W there exists a well defined birational map $q : W' \longrightarrow W$ called the monoidal transformation of W with center D; q is biregular on $W - D$; if W and D are nonsingular then so is W'; if D is a point then q is called the quadratic transformation with center D. If $q_1 : W_1 \longrightarrow W$, $q_2 : W_2 \longrightarrow W_1, \ldots, q_t : W_t \longrightarrow W_{t-1}$ is a sequence of monoidal transformations with nonsingular centers then the resulting birational map $q : W_t \longrightarrow W$ is called a composite monoidal transformation. Henceforth all monoidal transformations will be assumed to have nonsingular centers. A hypersurface H in a nonsingular variety W is said to have only normal crossings if for every point P of W there exist regular parameters $x_1, \ldots, x_n$ on W at P such that H is defined by $x_1 \ldots x_m = 0$ at P for some $m \leq n$.

Usually, after Embedded Resolution for n and before Resolution for n one proves the following:

Dominance (or removal of points of indeterminacy). Given two nonsingular projective models W and W^* of an n dimensional function field, there exists a composite monoidal transformation $W' \longrightarrow W$ such that W' dominates W^*.

Concerning Embedded Resolution I can definitely say that I have a proof for $n = 3$ in any characteristic; and this is the major step in the possible proof of Resolution for $n = 3$ (which I spoke of). Another dividend of this result is that I believe one can now prove the birational invariance of the arithmetic genus for dimension 3. In my proof of Embedded Resolution for $n = 3$ I draw heavily from Zariski's proof of the same result in characteristic zero which he gave in 1944 and also from the simplified proof of this which Zariski gave in a note in 1962 in the Rendiconti...Lincei.

§3. Peculiarities of nonzero characteristic

I shall now make various comments as to how the case $p \neq 0$ differs from the case $p = 0$ and what are the possible ways to make up for this difference. Let me say explicitly that it is not my purpose to find a proof which will be essentially new for $p = 0$. I am only trying to develop an algorithm or a calculus in $p \neq 0$ which will enable us to modify any given proof (of resolution) for $p = 0$ so that it will work also for $p \neq 0$.

(1) Binomial theorem. Algebraically speaking, the basic reason why the $p = 0$ proofs (of Zariski and Hironaka) fail for $p \neq 0$ is this:

Let

$$(Z + Y)^m = Z^m + a_1 Z^{m-1} Y + a_2 Z^{m-2} Y^2 + \dots + Y^m .$$

Then $a_1 \neq 0$ if $m \not\equiv 0(p)$, and $a_1 = 0$ if $m \equiv 0(p)$. More generally let

$$f(Z) = Z^m + f_1 Z^{m-1} + \dots + f_m$$

and

$$g(Z) = f(Z + Y) = Z^m + g_1 Z^{m-1} + \dots + g_m .$$

Then what is the relationship between the f_i and the g_j, i.e., which of the f_i affect a particular g_j and how much?

(2) <u>An example of</u> (1). Let

$$V : G(Z_1, \dots, Z_{n+1}) = 0$$

be an algebroid hypersurface in the $n + 1$ dimensional local space A_{n+1}. Let m be the multiplicity of V. Then upon making a linear transformation and invoking the Weierstrass Preparation Theorem we get

$$V : Z^m + g_1(Y_1, \dots, Y_n) Z^{m-1} + \dots + g_m(Y_1, \dots, Y_n) = 0$$

where $g_i(0, \dots, 0) = 0$. Let g be the product of those g_i which are non-zero. Let

$$H : g(Y_1, \dots, Y_n) = 0 .$$

Then H is an hypersurface in the n dimensional local space A_n. Apply Embedded Resolution to get a composite monoidal transformation $q: B \longrightarrow A_n$ such that $q^{-1}(H)$ has only normal crossings. Let $X_1, \ldots, X_n$ be suitable parameters at a point P in B. This amounts to substituting certain power series $u_1(X_1, \ldots, X_n), \ldots, u_n(X_1, \ldots, X_n)$ for $Y_1, \ldots, Y_n$ in $g(Y_1, \ldots, Y_n)$ so that

$$g(u_1(X_1, \ldots, X_n), \ldots, u_n(X_1, \ldots, X_n)) = g'(X_1, \ldots, X_n) X_1^{a(1)} \ldots X_n^{a(n)}$$

where $g'(0, \ldots, 0) \neq 0$ and $a(1), \ldots, a(n)$ are nonnegative integers. Then actually

$$g_i(u_1(X_1, \ldots, X_n), \ldots, u_n(X_1, \ldots, X_n)) = g_i'(X_1, \ldots, X_n) X_1^{a(i,1)} \ldots X_n^{a(i,n)}$$

for all i for which $g_i \neq 0$, where $g_i'(0, \ldots, 0) \neq 0$ and $a(i,j)$ are nonnegative integers. q induces $q^*: B \times A_1 \longrightarrow A_{n+1}$. Let V^* be the proper transform of V by q^{*-1}. Then at the point $(P, 0)$, V^* is given by

$$V^*: Z^m + \sum_{0 < i \leq m, g_i \neq 0} g_i'(X_1, \ldots, X_n) X_1^{a(i,1)} \ldots X_n^{a(i,n)} Z^{m-i} = 0$$

For the sake of simplicity let us suppose that $a(i,j) = 0$ whenever $j \neq 1$. Let b be the greatest integer such that $b \leq a(i,1)/i$ for all i for which $g_i \neq 0$. Then $a(i',1) - bi' < i'$ for some i'. Make the composite monoidal transformation given by: $Z = Z^* X_1^b$. Let V' be the proper transform of V^* under this transformation. Let Q be a point of V'. Then there exists a unique element d in k such that "$Z^* = d$ at Q" Let $Z' = Z^* - d$. Then $X_1, \ldots, X_n$, Z' are parameters at Q and at Q we have

$$V' : f(X_1, \ldots, X_n, Z') = 0$$

where

$$f(X_1, \ldots, X_n, Z')$$

$$= (Z' + d)^m + \sum_{0<i\leq m,\ g_i\neq 0} g_i'(X_1, \ldots, X_n) X_1^{a(i,1)-bi} (Z' + d)^{m-i} .$$

Let m' be the multiplicity of V' at Q. Since $a(i',1) - bi' < i'$ for some i', we get that if $d = 0$ then $m' < m$; a reduction! So now suppose that $d \neq 0$. At this point there are various essentially equivalent ways of arguing provided $m \not\equiv 0(p)$. For instance, following Hironaka, we can make the initial coordinate transformation $Z \longrightarrow Z - (1/m)g_1$ which will have the effect that the coefficient of Z^{m-1} will be zero, i.e., we will then have $g_1 = 0$ and hence

$$f(X_1, \ldots, X_n, Z') = Z'^m + dmZ'^{m-1} + \text{ terms of degree less than } m-1 \text{ in } Z'.$$

Then $m' < m$ because $dm \neq 0$. However if $m \equiv 0(p)$ then in the first place we cannot make the transformation $Z \longrightarrow (1/m)g_1$, and in the second place even if g_1 was zero to begin with we still cannot conclude that $m' < m$

because now $dm = 0$. We shall now make several observations.

(3). Peculiarity arises when the multiplicity is divisible by p.

(4) The most primitive case of the above pecularity is afforded by: $Z^p - g(Y_1, \ldots, Y_n) = 0$. The two dimension case (i.e., $n = 2$) of this was explicitly mentioned by Zariski in his 1950 address and there he pronounced it "intractable".

(5) It is not necessary to kill g_1 completely, i.e., it is enough to kill the terms of low degree in g_1. Because then $a(1,1) > b$ and hence again the coefficient of Z'^{m-1} in $f(Z')$ will be a unit. In any case if we have $a(1,1) > b$ then we are all right. In other words, the X_1-value of g_1 should be big enough compared to the X_1-values of the other g_i. If $m \equiv 0(p)$ then of course g_1 does not play the dominant role. But it turns out that even then we would be in a reasonable shape if say

$$(X_1\text{-value of } g_i) \geq (i/m)(X_1\text{-value of } g_m) \text{ for } i = 1, \ldots, m .$$

It can be shown that if the above inequality fails for some i then X_1 must split in the covering $V' \longrightarrow$ (space of $X_1, \ldots, X_n$), i.e., X_1 must split in the field extension given by $f(X_1, \ldots, X_n, Z')$. So one should try to arrange that X_1 does not split.

(6). In the general case, i.e., when one does not assume $a(i,j) = 0$ for $j \neq 1$, one tries to arrange matters so that for any $i \neq i^*$ either g_i divides g_{i^*} or g_{i^*} divides g_i, i.e., either $a(i,j) \leq a(i^*,j)$ for all j or $a(i^*, j) \leqq a(i, j)$ for all j. This really means that instead of only applying "Embedded Resolution" we also invoke "Dominance". In this general case one should accordingly try to arrange that each of the X_j which occur with a positive exponent does not split in the field extension given by f.

(7) Instead of killing g_1, Zariski used differentiation arguments. But then after all the binomial theorem and differentiation are in essence one and the same thing.

In §4 and §5 I shall further elucidate observations (5) and (4) respectively.

§4. Nonsplitting

Let W be a nonsingular projective algebraic variety of dimension n and let V be the normalization of W in a finite algebraic separable extension of the function field of W, i.e., we have a covering map $V \longrightarrow W$. Let D be the branch locus on W. By Embedded Resolution we can find a composite monoidal transformation $q: W' \longrightarrow W$ such that $q^{-1}(D)$ has only normal crossings. Let $h: V' \longrightarrow W'$ be the corresponding covering map and let D' be the branch locus on W'. Then $D' \subset q^{-1}(D)$. It can then be shown if $p = 0$ (or more generally if $V' \longrightarrow W'$ is a tame covering) then the irreducible components of $q^{-1}(D)$ do not split locally on

V', i.e., if $P' \in V'$ lies above $Q' \in W'$ and E is any irreducible component of $q^{-1}(D)$ at Q' then only one locally irreducible component of $h^{-1}(E)$ passes through P'. In other words, if $f(Z) = Z^m + g_1 Z^{m-1} + \dots + g_m$ is a local equation of the covering $P' \longrightarrow Q'$ and $X_1, \dots, X_n$ are parameters at Q' such that $q^{-1}(D) \subset (X_1 \cdots X_n = 0)$ then for $i = 1, \dots, n$ we have that the valuation at Q' given by X_i does not split in the field extension given by $f(Z)$. Thus what was achieved by Hironaka by killing g_1 and by Zariski by using *differentiation* arguments can also be achieved by simplifying the branch locus. The idea of simplifying the branch locus to resolve the singularities of V was actually used by Jung for $n = 2$ and $k =$ the field of complex numbers, and it was also proposed by Zariski (1954: Bulletin des Sciences Mathematiques) as a possible method of resolution of singularities for all n when $p = 0$; also Zariski used this idea in his Lincei note cited in §2. Both of them used this only to have a nice structure for the local ring of P' and not for the nonsplitting business. However, we thus see that this Jungian method of simplifying the branch locus (i.e., transforming the discriminant into a monomial times a unit) and the Zariski-Hironaka method of transforming the coefficients into monomials times units are in essence very closely related, although they may not appear so at first sight.

Actually in 1953, Zariski had suggested to me to study Jung's method and to see whether it could be used for resolution of singularities of a surface in $p \neq 0$. At that time I ended up by showing that in $p \neq 0$: the local Galois group above a simple point of the branch locus can be unsolvable, and a point lying above a simple point of the branch locus can be singular, and hence Jung's method cannot be used. The examples were published in 1955 in the American Journal where it was shown that they can occur only for nontame coverings. There I also showed that although the nonsplitting holds for tame coverings, in general it does not, and I went on to comment that: this "local splitting of a simple branch variety by itself" is the real reason behind the peculiarity in $p \neq 0$. Later on in a 1957 paper in the American Journal I exploited a similar splitting of a branch point on a curve to get results like the following: every curve in $p \neq 0$ can be projected onto the projective line so as to have only one branch point. On the other hand, in a series of papers published in the American Journal in 1959-1960 I used the nonsplitting for tame coverings to study the tame fundamental group of an algebraic variety.

Now after eight years the circle is completed. Namely, it turned out that (for a covering of a surface where the covering degree is either not divisible by p or is a power of p) if we keep applying quadratic transformations, even after the stage when the branch locus has only normal crossings, then eventually we shall reach a stage when we have nonsplitting; and what is more important is that we can reach a stage which is stable, i.e., when the nonsplitting is not destroyed by applying more quadratic transformations; needless to remark that the number of quadratic transformations required to achieve such a stable nonsplitting stage depends on the given covering. Moreover, in the end we reach a Jungian situation after all. This realization was forced upon me by working on the arithmetical case in which no single method seems to work by itself. The arriving at a stable nonsplitting stage is also the main novel aspect of my proof of Embedded Resolution for $n = 3$, i.e., for surfaces.

All this leads me to pose the following conjectural supplement to Embedded Resolution.

Supplement 1. Let W be a nonsingular projective algebraic variety of dimension n, let V be the normalization of W in a finite algebraic separable extension of the function field of W, and let D be the branch locus on W for the covering $V \longrightarrow W$. You may assume that D has only normal crossings. Find a composite monoidal transformation $q : W' \longrightarrow W$ such that nonsplitting holds for $q^{-1}(D)$ relative to the corresponding covering $V' \longrightarrow W'$. Do this in some stable sense.

It is proposed to use this as an inductive step in the general resolution problem.

§5. Units cannot be neglected

Let us now consider the primitive case

$$V : Z^p - g(Y_1, \ldots, Y_n) = 0 .$$

The nonsplitting business clearly has no bearing on this. By Embedded Resolution we can achieve

$$g(Y_1, \ldots, Y_n) = g'(X_1, \ldots, X_n) X_1^{a(1)} \ldots X_n^{a(n)}$$

where $g'(0, \ldots, 0) \neq 0$. If at least one of the $a(i)$ is not divisible by p then we can do something. But if $a(i) \equiv 0(p)$ for all i then upon making a composite monoidal transformation we get

$$V' : Z'^p - g^*(X_1, \ldots, X_n) = 0$$

where

$$g^*(X_1, \ldots, X_n) = g'(X_1, \ldots, X_n) - g'(0, \ldots, 0).$$

So we achieved nothing because the order of $g^*(X_1, \ldots, X_n)$ may even be greater than the order of $g(Y_1, \ldots, Y_n)$. In other words, in $p \neq 0$ we cannot neglect the unit $g'(X_1, \ldots, X_n)$. A similar situation prevails for

$$V : Z^{p^u} - g(Y_1, \ldots, Y_n) = 0 .$$

Thus we are led to another conjectural supplement to Embedded Resolution; this one cannot be formulated completely geometrically, i.e., we cannot talk of a hypersurface but we must actually deal with a power series, because we are now not interested in a principal ideal but in a specific power series.

<u>Supplement</u> 2. Let $m = p^u$ where u is a positive integer. Let g be an element in the power series ring $k[[Y_1, \ldots, Y_n]]$ such that $g \notin k[[Y_1^m, \ldots, Y_n^m]]$. Find a composite monoidal transformation $q : B \longrightarrow A_n$, where A_n is the local space of $Y_1, \ldots, Y_n$, such that at any $P \in B$ there exist suitable parameters $X_1, \ldots, X_n$ such that upon considering g as an element in $k[[X_1, \ldots, X_n]]$ we have that

$$g = h^m + (X_1^{a(1)} \ldots X_n^{a(n)})^m g'$$

where $a(1), \ldots, a(n)$ are nonnegative integers and h and g' are elements in $k[[X_1, \ldots, X_n]]$ such that $0 < (\text{order of } g') < m$.

Actually this is not entirely satisfactory because it is not a stable situation. One must ask for a stable situation. Here I shall not pursue this matter further because things would get too technical.

As such the primitive case may not occur in practice because we can choose a separating transcendence basis, etc. However, in the separable case

$$Z^p + g_1(Y_1,\ldots,Y_n)Z^{p-1} + \ldots + g_p(Y_1,\ldots,Y_n)$$

the nonsplitting business will help only to see to it that $g_1,\ldots,g_{p-1}$ do not intefore too much. The game is still to be played with g_p. In other words, it is proposed that:

(general case) = (primitive case) + (nonsplitting).

Anyway, this is how I carry out things for surfaces.

§6. Resolution for coverings

I shall conclude by mentioning a more general resolution problem which is of interest in itself and some form of which may very well be useful in an inductive set up for the original resolution problem

(1). Give an intrinsic definition of a Jungian local domain, i.e., of a normal local domain which in case of characteristic zero can be projected onto a regular local domain so that the branch locus has a normal crossing; (for dimension 2 I have done this in a forthcoming paper).

(2). Given a function field K and a finite algebraic separable extension L of K, does there exist a nonsingular model of K whose normalization in L is Jungian?

(3). Given a function field K and a finite algebraic separable extension L of K, does there exist a Jungian model of K whose normalization in L is nonsingular?

I shall only remark that (3) is nontrivial even when the characteristic is zero and we require the model of K to be only normal.

EQUIVALENCES AND DEFORMATIONS OF ISOLATED SINGULARITIES

by H. Hironaka

When I speak of deformations of isolated singular points on algebraic schemes, the basic setup is as follows:

$\pi : X \to Y$ <u>is a morphism of schemes</u>, Y <u>is a noetherian scheme</u>, π <u>is of finite type and flat</u>, $\varepsilon : Y \to X$ <u>is a morphism such that</u> $\pi \circ \varepsilon$ = <u>identity</u>, π <u>is smooth on</u> $X - \varepsilon(Y)$, <u>and all the fibres</u> $X_y = \pi^{-1}(y)$, $y \in Y$, <u>are reduced and equidimensional</u> (i.e., all the irreducible components of X_y have the same dimension). Here the word "<u>smooth</u>" means that if x is any point of $X - \varepsilon(Y)$ then (assuming that Y is affine, say $\mathrm{Spec}(A)$, without any loss of generality) π is decomposed into an étale morphism from a neighborhood of x in X to $\underline{\mathrm{Spec}}(A[t_1, \dots, t_n])$ and the projection from this spectrum to $Y = \mathrm{Spec}(A)$, where n = the dimension of the fibres of π. It follows from the assumptions, that the fibre X_y for each $y \in Y$ is non-singular, except for the possible singularity at $\varepsilon(y) \in X_y$. Thus we have a family of algebraic schemes with (possible) isolated singular points, $\{(X_y, \varepsilon(y))\}$, which are parametrized by the points of Y. If Y is a non-singular curve, then the flatness of π means simply that every irreducible component of X is surjectively mapped to Y. In general, it implies that all the fibres X_y, $y \in Y$, have the same dimension and that every irreducible component of X is surjectively mapped to a connected component of Y.

In this basic setup, we do not lose too much by taking "algebraic varieties, say, over an algebraically closed field" instead of "schemes". However, we need to consider some other "derived" setups, such as <u>truncations</u> and <u>completions</u> of the given family, in which "schemes with nilpotent elements in the structure sheaves" and "formal schemes" are involved.

Suppose a family (of isolated singularities), (π, X, Y, ε), is given as above. Let $\underline{I}$ be the ideal sheaf of the subscheme $\varepsilon(Y)$ in $\underline{O}_X$. We write then $X_{(\nu)}$ for the subscheme of X defined by the ideal sheaf $\underline{I}^{\nu+1}$, where ν is any non-negative integer. $X_{(0)} = \varepsilon(Y)$ and all the $X_{(\nu)}$, $\nu \geq 0$, have the same underlying topological space. The structure sheaf of $X_{(\nu)}$ is $\underline{O}_X/\underline{I}^{\nu+1}$ (restricted to its support). We have a canonical immersion $X_{(\nu)} \rightarrow X_{(\mu)}$ for $\mu > \nu$, and we call the limit space $\hat{X} = \varinjlim_{\nu} X_{(\nu)}$ the <u>$\underline{I}$-adic completion</u> of X. The structure sheaf of this "<u>formal scheme</u>" $\hat{X}$ is $\varprojlim_{\nu} \underline{O}_X/\underline{I}^{\nu+1}$. The morphism $\pi: X \rightarrow Y$ (resp. $\varepsilon: Y \rightarrow X$) induces morphisms $\pi_{(\nu)}: X_{(\nu)} \rightarrow Y$ and $\hat{\pi}: \hat{X} \rightarrow Y$ (resp. $\varepsilon_{(\nu)}: Y \rightarrow X_{(\nu)}$ and $\hat{\varepsilon}_{(\nu)}: Y \rightarrow \hat{X}$). The "derived" family $(\pi_{(\nu)}, X_{(\nu)}, Y, \varepsilon_{(\nu)})$ (resp. $(\hat{\pi}, \hat{X}, Y, \hat{\varepsilon})$) will be called <u>the ν-th truncation</u> (resp. <u>the completion</u>) of the given (π, X, Y, ε).

In what follows, the main theme is to compare a family (π, X, Y, ε) with another $(\pi', X', Y, \varepsilon')$ (likewise, their truncations, or their completions), where the parameter space Y is the same for all. By abuse of language, I shall say, for instance, that

$$\varphi_{(\nu)}: (\pi_{(\nu)}, X_{(\nu)}, Y, \varepsilon_{(\nu)}) \rightarrow (\pi'_{(\nu)}, X'_{(\nu)}, Y, \varepsilon'_{(\nu)})$$

is a morphism (or an isomorphism) <u>within a neighborhood of a point</u> $y \in Y$, when $\varphi_{(\nu)}$ is meant to be a morphism (or an isomorphism) from

$\pi_{(\nu)}^{-1}(U)$ to $\pi'^{-1}_{(\nu)}(U)$ such that $\pi'_{(\nu)} \circ \varphi_{(\nu)} = \pi_{(\nu)}$, where U is a certain neighborhood of the point y in Y.

Theorem 1. Let (π, X, Y, ε) be a family of isolated singular points (in the sense described above). Let y be a point of Y. Then there exists a pair of integers (t, r), $t \geq 1$ and $r \geq 0$, which has the following properties

(I) Let ν be an integer not less than t, and $(\pi', X', Y, \varepsilon')$ any family of isolated singular points. Suppose $\dim(X_y) = \dim(X'_y)$ and there is given an isomorphism $\varphi_{(\nu)}: (\pi'_{(\nu)}, X'_{(\nu)}, Y, \varepsilon'_{(\nu)}) \xrightarrow{\approx} (\pi_{(\nu)}, X_{(\nu)}, Y, \varepsilon_{(\nu)})$ within a neighborhood of y. Then the isomorphism

$\varphi_{(\nu-r)}: (\pi'_{(\nu-r)}, X'_{(\nu-r)}, Y, \varepsilon'_{(\nu-r)}) \xrightarrow{\approx} (\pi_{(\nu-r)}, X_{(\nu-r)}, Y, \varepsilon_{(\nu-r)})$, induced by $\varphi_{(\nu)}$, extends to an isomorphism $\hat{\varphi}: (\hat{\pi}', \hat{X}', Y, \hat{\varepsilon}') \xrightarrow{\approx} (\hat{\pi}, \hat{X}, Y, \hat{\varepsilon})$ within a neighborhood of y.

(II) If $h: \tilde{Y} \to Y$ is any morphism of noetherian schemes and $\tilde{y}$ is any point of $\tilde{Y}$ such that $h(\tilde{y}) = y$, then the pair of integers (t, r) has the same property as (I) for the family $(\tilde{\pi}, \tilde{X}, \tilde{Y}, \tilde{\varepsilon})$ obtained from (π, X, Y, ε) by the base extension h, and for the point $\tilde{y}$ of $\tilde{Y}$, where $\tilde{X} = X \times_Y \tilde{Y}$, $\tilde{\pi} = \pi \times_Y \tilde{Y}$, $\tilde{\varepsilon} = \varepsilon \times_Y \tilde{Y}$.

In particular, I shall consider the case in which X is an algebraic k-scheme, with a field k, and Y is a geometric point of X with value in k (or, a k-rational point). In this case, the theorem asserts, roughly speaking, that the analytic structure of the isolated singular point Y of X is determined by $\dim X$ and by the structure of the truncated local algebra $\underline{O}_Y/\underline{m}_Y^{t+1}$, where $\underline{O}_Y$ = the local ring of X at Y and $\underline{m}_Y$ the maximal ideal of $\underline{O}_Y$.

Definition 1. For a family of isolated singular points (π, X, Y, ϵ) and a point y of Y, I call (t, r) a pair of TR - indices of (π, X, Y, ϵ) at the point y, if it has the properties (I) and (II) stated in Theorem 1. In particular, when an algebraic k - scheme X is given with an isolated singular point y (with value in k), (t, r) will be called a pair of TR - indices of (X, y) (or, that of the isolated singular point y of X).

Note that if (t, r) is a pair of TR - indices of (π, X, Y, ϵ) then so is every pair of integers $(\overline{t}, \overline{r})$ such that $\overline{t} \geq t$ and $\overline{r} \geq r$. It can be proved that the integer r can be zero only if $\epsilon(y)$ is a simple point of X_y.

The theorem has an obvious "complex - analytic" analogue, in which X and Y are complex - analytic spaces with holomorphic maps π and ϵ. In the complex - analytic case, one can find a proof of the theorem (for an isolated singular point $Y = y$) in HIRONAKA - ROSSI, [3], which is based on desingularization techniques (HIRONAKA, [2]; especially, Corollary 1, p. 153, § 7, Chap. 0) and infinitesimal calculus due to Grothendieck - Grauert. In this case, Grauert's normal projection method ([1], Satz 5, p. 359) gives a stronger conclusion in which the extended isomorphism $\hat{\phi}$ of the theorem is holomorphic (or, to be precise, $\hat{\varphi}$ is the formalization of a biholomorphic map from a neighborhood of $\epsilon'(y)$ in X' to a neighborhood of $\epsilon(y)$ in X) It was then pointed out by M. Artin that the normal projection method provides an étale equivalence (which implies a holomorphic one in the complex case) instead of the formal equivalence $\hat{\phi}$. On the other hand, I found a new proof of the theorem (which works in the above - stated generality; for instance, in any characteristic case) and by these means, I obtained a theorem of the following type.

__Theorem__ 2. __Let__ (π, X, Y, ε) __and__ $(\pi', X', Y, \varepsilon')$ __be families of isolated singular points,__ __and__ y __a point of__ Y. __Suppose there is given an isomorphism__ $\hat{\varphi}: (\hat{\pi}', \hat{X}', Y, \hat{\varepsilon}') \xrightarrow{\approx} (\hat{\pi}, \hat{X}, Y, \hat{\varepsilon})$ __within a neighborhood of__ y __in__ Y. __Let__ ν __be any positive integer.__ __Then,__ __within a neighborhood of__ y, __there exist etale morphisms__ $\lambda: (\tilde{\pi}, \tilde{X}, Y, \tilde{\varepsilon}) \to (\pi, X, Y, \varepsilon)$ __and__ $\lambda': (\tilde{\pi}', \tilde{X}', Y, \tilde{\varepsilon}') \to (\pi', X', Y, \varepsilon')$, __which induce isomorphisms of completions,__ __and there exists an isomorphism__ $\tilde{\varphi}: (\tilde{\pi}', \tilde{X}', Y, \tilde{\varepsilon}') \xrightarrow{\approx} (\tilde{\pi}, \tilde{X}, Y, \tilde{\varepsilon})$ __which induces the same isomorphism__ $\varphi_{(\nu)}: (\pi'_{(\nu)}, X'_{(\nu)}, Y, \varepsilon'_{(\nu)}) \xrightarrow{\approx} (\pi_{(\nu)}, X_{(\nu)}, Y, \varepsilon_{(\nu)})$ __as the given__ $\hat{\varphi}$ __does.__

In this theorem, étale morphisms are meant to be those of finite type. In my proof, this theorem and the preceding one are proven simultaneously. It should be interesting to find a direct proof of the second theorem and to look into the question of whether it is essential or not to assume the smoothness of $\pi: X \to Y$ for __all__ points of $X - \varepsilon(Y)$. (Note that this smoothness assumption is essential in the first theorem.)

Let us now go back to investigate further the notion of TR - indices of a family of isolated singular points.

__Theorem__ 3. __Let__ (π, X, Y, ε) __be a family of isolated singluar points and__ y __a point of__ Y. __Let__ (t, r) __be a pair of__ TR - __indices of__ (π, X, Y, ε) __at__ y. __Then there exists a neighborhood__ U __of__ y __in__ Y __such that for every point__ z __of__ U, (t, r) __is a pair of__ TR - __indices for the isolated singular points__ $(X_z, \varepsilon(z))$, __where__ $X_z = \pi^{-1}(z)$.

The converse of the theorem is far from being true. Namely, let us introduce the notion of TR - indices of all "near-by" fibres of a family (π, X, Y, ε) at a point y of Y. This is any pair of integers $(\bar{t}, \bar{r})$, $\bar{t} \geq 1$ and $\bar{r} \geq 0$, such that there exists a neighborhood U of y in Y such that $(\bar{t}, \bar{r})$ is a pair of TR - indices of the fibres $(X_z, \varepsilon(z))$ for all $z \in U$. Then the claim is that $(\bar{t}, \bar{r})$ is not in general a pair of TR - indices of (π, X, Y, ε) at y.

The difference between the above two notions of TR - indices for a family, namely (t, r) and $(\bar{t}, \bar{r})$, can be seen in the following two theorems, the first of "affirmative nature" and the second of "negative nature".

Let us start with a fixed isolated singular point of an algebraic k - scheme (X_0, x_0). Let us then consider various families of isolated singular points (π, X, Y, ε) with center (X_0, x_0); that is to say, there is a specified point y_0 of Y and an isomorphism $\hat{\varphi}_0 : (\hat{X}_0, x_0) \xrightarrow{\approx} (\hat{X}y_0, \varepsilon(y_0))$, where $\hat{X}_0$ (resp. $\hat{X}y_0$) denotes the completion of X_0 (resp. Xy_0) by the powers of the maximal ideal at x_0 (resp. the same at $\varepsilon(y_0)$).

Theorem 4. Given an isolated singular point (X_0, x_0), there exists a pair of integers $(\bar{t}, \bar{r})$ such that if (π, X, Y, ε) is any family of isolated singular points with center $(\hat{X}_0, x_0) \xrightarrow{\approx} (\hat{X}y_0, \varepsilon(y_0))$, then there exists a neighborhood U of y_0 in Y and $(\bar{t}, \bar{r})$ is a pair of TR - indices for all fibres $(Xy, \varepsilon(y))$ with $y \in U$.

Theorem 5. Suppose (X_0, x_0) is an isolated singular point of a complete intersection X_0, i.e., there exists a local imbedding of X_0 in an affine N - space so that the local ideal of X_0 at x_0 is generated by $(N - \dim X_0)$ elements. Suppose x_0 is not a simple point of X_0. Then, for every pair of integers (t, r), there exists a family of isolated singular points (π, X, Y, ε) with center $(\hat{X}_0, x_0) \xrightarrow{\approx} (\hat{X}y_0, \varepsilon(y_0))$, such that (t, r) is not a pair of TR-indices of (π, X, Y, ε) at y_0.

Let us remark that if (π, X, Y, ε) is a family of isolated singular points and Xy is a complete intersection at $\varepsilon(y)$ for some $y \in Y$, then there exists a neighborhood U of y in Y so that all the fibres X_z is a complete intersection at $\varepsilon(z)$ for all $z \in U$. In fact, X itself is a complete intersection locally at $\varepsilon(y)$ in Spec $(A[t_1, \ldots, t_N])$ with independent variables t_j $(1 \leqq j \leqq N)$, where $y \in$ Spec $(A) \subseteq Y$.

In dealing with complete intersections, there is another point that makes "deformation theory" simpler than the general case. Namely, let $\pi : X \to Y$ be any flat morphism of finite type, say with a regular noetherian scheme Y, and $\varepsilon : Y \to X$ is a section, i.e., a morphism such that $\pi \cdot \varepsilon$ = identity. Suppose that for every point $y \in Y$, the fibre Xy is reduced and equidimensional and $\varepsilon(y)$ is an isolated singular point of Xy. Notice that such (π, X, Y, ε) is a "family" of isolated singular points (in the sense of this paper) under only one additional condition that $X - \varepsilon(Y)$ is smooth over Y. This condition means that $Xy - \varepsilon(y)$ is non-singular for all $y \in Y$. Now the point is that if all the fibres Xy are complete intersections then we can always modify (π, X, Y, ε) into another $(\bar{\pi}, \bar{X}, Y, \bar{\varepsilon})$, satisfying the additional condition, in such a way that there exists an isomorphism $(\bar{\pi}_{(\nu)}, \bar{X}_{(\nu)}, Y, \bar{\varepsilon}_{(\nu)}) \xrightarrow{\approx} (\pi_{(\nu)}, X_{(\nu)}, Y, \varepsilon_{(\nu)})$ where (ν, μ) for some μ is a pair of TR-indices for all the isolated singular points $(Xy, \varepsilon(y))$, $y \in Y$.

Let us furthermore remark that the existence of $(\bar{t}, \bar{r})$ in Theorem 4 suggests that:

The totality of all near-by isolated singularities of a given one is "of finite type" in some algebro - geometric sense.

The meaning of such a statement can be made very precise in the case of complete intersections. Namely, let me introduce the notion of "quasi - equivalent" families. This is as follows. Let (π, X, Y, ε) and $(\pi', X', Y, \varepsilon')$ be two families of isolated singular points, having the same parameter space Y. Then I say that they are quasi - equivalent, if there exists an isomorphism $(\pi'_{(\nu)}, X'_{(\nu)}, Y, \varepsilon'_{(\nu)}) \xrightarrow{\approx} (\pi_{(\nu)}, X_{(\nu)}, Y, \varepsilon_{(\nu)})$, where (ν, μ) for some μ is a pair of TR - indices for all the fibres $(Xy, \varepsilon(y))$ and $(X'y, \varepsilon'(y))$, $y \in Y$. Now, I claim that Theorem 4 implies, for instance, that, if (X_0, x_0) is a complete intersection, then there exists a family of isolated singular points $(\pi^*, X^*, Y^*, \varepsilon^*)$ with center $(\hat{X}_0, x_0) \xrightarrow{\approx} (\hat{X}^*y_0, \varepsilon^*(y_0))$ which induces every other family of isolated singular points with the same center, "up to quasi - equivalences" within some neighborhoods of the center.

Theorem 5, on the other hand, implies that there exists no family of isolated singular points with center $(\hat{X}_0, x_0)$ which induces every other family of isolated singular points with the same center, "up to formal equivalences" (or, up to isomorphisms of completions) within suitable neighborhoods of the center.

These facts attract me towards the question of finding "reasonable and (or) significant restrictions" to be imposed on families involved, which enable us to construct "a local universal family up to formal equivalences". For instance, the question leads me to the notion of "equi - singularity". A theory of equi - singularity has been dug up by Zariski and is gradually showing up its clear face in some special cases. But it seems that a full and complete theory is at present utterly out of sight.

Zariski's theory of equi - singularity is strictly concerned with "hypersurfaces". Here I like to propose a notion of equi - singularity which applies even to non - hypersurface cases, although the characteristic of the base field is required to be zero as in Zariski's case.

Definition 2. Let X be an algebraic scheme over a field k of characteristic zero. Assume that X is reduced and equi - dimensional. Let Y be a non - singular irreducible subscheme of X. Let y be a point of Y. Then I say that X is equi - singular along Y at the point y, if (replacing X, and Y accordingly, by a neighborhood of y) there exists a morphism $\pi: X \to Y$ which has the following properties:

i) (π, X, Y, ε) is a family of isolated singular points (in the sense described at the very beginning), where $\varepsilon: Y \to X$ denotes the canonical immersion, and

ii) let $\mathcal{J}_0$ be the sheaf of Jacobian ideals of (π, X, Y, ε) (see below) and $\mathcal{I}$ the sheaf of ideals defining Y on X. Take the sheaf of product ideals, $\mathcal{J} = \mathcal{J}_0 \mathcal{I}$, on X, and the composition $\tilde{h}: \tilde{X} \to X$ of the birational blowing - up of X by the ideals $\mathcal{J}$ and the normalization of the blown - up scheme. Let $\tilde{\mathcal{J}} = \mathcal{J} \underline{O}_{\tilde{X}}$, the ideal sheaf on $\tilde{X}$ generated by $\mathcal{J}$. Then $\underline{O}_{\tilde{X}} / \tilde{\mathcal{J}}$ is flat over $\underline{O}_Y$, or, the subscheme $\tilde{Y}$ of $\tilde{X}$ defined by $\tilde{\mathcal{J}}$ is flat over Y with reference to the morphism $\pi \circ \tilde{h}$.

The sheaf of Jacobian ideals of (π, X, Y, ε) is defined as follows:

For each point $y \in Y$, by replacing X by a suitable neighborhood of $\varepsilon(y)$ (and accordingly Y), I assume that there exists an imbedding $p: X \rightarrow \underline{\mathrm{Spec}}\,(A[T_1, T_2, \ldots, T_N])$, where $Y = \underline{\mathrm{Spec}}\,(A)$ and T_j $(1 \leqq j \leqq N)$ are independent variables, such that $\pi = (\text{projection}) \circ p$ and the ideal of $(p \circ \varepsilon)(Y)$ is generated by $T_1, T_2, \ldots, T_N$. Then the sheaf of Jacobian ideals $\mathfrak{J}_0$ on X is generated by $(N - \underline{\dim}\, X_y) \times (N - \underline{\dim}\, X_y)$ - minors of the Jacobian matrix $\partial(f_1, \ldots, f_m)/\partial(T_1, \ldots, T_N)$, where $(f_1, \ldots, f_m)$ is a base of the ideal of $p(X)$ in $A[T]$. It is important that the sheaf $\mathfrak{J}_0$ is independent of the choice of imbedding p.

Remark. If the condition i) of Definition 2 is satisfied, then the condition ii) is equivalent to the following:

ii*) $\tilde{\mathfrak{J}}^{\nu}/\tilde{\mathfrak{J}}^{\nu+1}$ is flat over $\underline{O}_Y$ for all integers $\nu \geqq 0$.

Moreover, if $\underline{\dim}\, Y = 1$, then it is equivalent to:

ii**) Every irreducible component of $\tilde{h}^{-1}(\varepsilon(Y))$ is mapped onto a connected component of Y by $\pi \circ \tilde{h}$.

The following theorem suggests the possibility that the totality of all small families of isolated singular points with a given center can be derived from a certain universal family so long as they are subject to some equi-singularity condition.

Theorem 6. Let (X_0, x_0) be an isolated singular point. Then there exists a pair of integers (t, r) such that for every family of isolated singular points (π, X, Y, ε) with center $(\hat{X}_0, x_0) \cong (\hat{X}_{y_0}, \varepsilon(y_0))$, (t, r) is a pair of TR - indices of (π, X, Y, ε) at y_0 provided (π, X, Y, ε) satisfies the condition ii) of Definition 2.

It seems to me that Theorem 6 remains true if one replaces the condition ii) of Definition 2 by some reasonably weaker condition.

The condition ii) of Definition 2 has some geometric and topological significances. From now on, this condition will be referred to as "condition (ES)". I take now the case in which the base field k is the complex number field $\mathbb{C}$. In the place of "schemes", I shall take "complex-analytic varieties". Notice that the "condition (ES)" has an obvious analogue in the complex-analytic case. Let (X_0, x_0) be an isolated singular point, where X_0 is now a complex-analytic variety (reduced and equi-dimensional Suppose we have a local imbedding $p_0: X_0 \to \mathbb{C}^N$, $\mathbb{C}^N$ = the complex number space of dimension N. For simplicity, assume that $p_0(x_0) = 0$, the origin. Let $T_x(X_0)$ be the complex tangent space of X_0 at $x \in X_0$, $\neq x_0$, which is realized as a linear subspace of $\mathbb{C}^N$ in a natural manner. If $u = (u_1, u_2, \dots, u_N)$ and $v = (v_1, v_2, \dots, v_N)$ are two vectors in $\mathbb{C}^N$, then $u \cdot v$ denotes the inner product $\sum_{i=1}^N u_i \cdot \overline{v_i}$. For a point $x \in X_0$, let $\overrightarrow{0x}$ denote the vector in $\mathbb{C}^N$ which ends at x (and, as always, starts from the origin). Consider the following real-valued function of $x \in X_0 - \{x_0\}$,

$$\tau(p_0; x) = \max_{\substack{v \in T_x(X_0) \\ v \neq 0}} \left\{ \frac{|v \cdot \overrightarrow{0x}|}{|v|\,|\overrightarrow{0x}|} \right\}$$

where $|v|$ = the length of $v = \sqrt{\sum_{i=1}^N |v_i|^2}$. Then, Whitney proves:

<u>Theorem</u> 7.

$$\lim_{x \to 0} \tau(p_0; x) = 1 .$$

By this theorem, if we set $\tau(p_0; x_0) = 1$, then $\tau(p_0; x)$ becomes a continuous function on X_0. Suppose there is given a family of isolated singular points (π, X, Y, ε). Then, at least locally, one can find an imbedding of the form $p: X \to Y \times \mathbb{C}^N$ such that $\pi = (\text{projection}) \circ p$ and $(p \circ \varepsilon)(Y) = Y \times 0$. Let me say that such an imbedding p is permissible. Pick one permissible imbedding p. Then, for each point $y \in Y$, p induces an imbedding $p_y : Xy \to \mathbb{C}^N$ $(= y \times \mathbb{C}^N)$. For this induced imbedding p_y, I have a continuous real-valued function $\tau(p_y; x)$, $x \in Xy$. Let us define a real-valued function on X as follows:

$$\tau(p; x) = \tau(p_y; x) \quad \text{if} \quad x \in Xy, \quad y \in Y.$$

I shall call such a function $\tau(p; x)$ a W-function for the family of isolated singular points (π, X, Y, ε). This function depends upon the choice of imbedding p. I shall say that the W-function $\tau(p; x)$ is associated with the imbedding p. One can prove that, given a family of isolated singular points (π, X, Y, ε), if a W-function associated with a permissible imbedding is continuous on X, then the same holds for every permissible imbedding. In view of this fact, we can speak of the continuity of W-function for a given (π, X, Y, ε) without asking if X admits a global imbedding $X \to Y \times \mathbb{C}^N$ for some N, because X admits a permissible imbedding at least locally at every point of $\varepsilon(Y)$.

In view of the theorem of Whitney (or, some other reasons), given an isolated singular point (X_0, x_0) with an imbedding $p_0 : X_0 \to \mathbb{C}^N$ with $p_0(x_0) = 0$, one can find a real number $\rho > 0$ such that if $0 < \varepsilon < \rho$, then X_0 is transversal to the sphere in $\mathbb{C}^N$ with center 0 and with radius ε; this sphere S_ε has real dimension $2N-1$. Such a real number ρ will be called a permissible radius of (X_0, x_0) with p_0. Hence $W_\varepsilon = X_0 \cap S_\varepsilon$ is a manifold of dimension $2n-1$, where $n = \dim X_0$. If we identify S_ε with a standard $(2N-1)$-sphere S, then the imbedding $W_\varepsilon \to S$ is a differentiable isotopy in terms of the parameter ε $(0 < \varepsilon \leqq \rho)$. Let us simply write W for W_ε, and the isotopy class (W, S) is the "topology" of the singular point (X_0, x_0) with imbedding p_0. A family (π, X, Y, ε) with a permissible imbedding $p : X \to Y \times \mathbb{C}^N$ will be said to be topologically stable if for every point $\bar{y}$ of Y, there exists a neighborhood U of $\bar{y}$ in Y and a positive real number ρ such that ρ is a permissible radius of $(Xy, \varepsilon(y))$ with the induced imbedding $p_y : Xy \to \mathbb{C}^N$ for all $y \in U$.

I can prove:

Theorem 8. Given a family of isolated singular points (π, X, Y, ε) with a non-singular irreducible Y and with a permissible imbedding $p : X \to Y \times \mathbb{C}^N$,

(I) the condition (ES) $\Rightarrow$ the continuity of W-function $\Rightarrow$ the topological stability;

(II) if $\dim Xy = 1$ for $y \in Y$, then the continuity of W-function $\Rightarrow$ the order of singularity $\delta(Xy, \varepsilon(y))$ is constant for $y \in Y$. ($\delta(Xy, \varepsilon(y)) = \dim_C \tilde{O}/O$, where O = the local ring of Xy at $\varepsilon(y)$ and $\tilde{O}$ = the integral closure of O in the total ring of fractions of O, both being viewed as vector spaces over C.);

(III) if $\dim Xy = 1$ for $y \in Y$ and $N = 2$ (i.e., the case of plane curves), there is a complete equivalence of various conditions, namely, the condition (ES) $\Rightarrow$ the continuity of W-function $\Rightarrow$ the topological stability $\Rightarrow$ the condition (ES).

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EQUISINGULARITY AND RELATED QUESTIONS OF

CLASSIFICATION OF SINGULARITIES.

by

O. Zariski

§1. The elusive idea of equivalent singularities.

Ideally, a complete theory of equivalence of singularities must give a precise meaning to a statement such as this: "the singularity which a given variety V has at a given point P is "the same" as the singularity which another given variety V' has at a given point P'". In addition, the theory must include a number of criteria of equivalence, whether algebraic or algebro-geometric in nature; or topological, if we are dealing with the complex domain. Naturally, one will impose some restrictions on the ground field k, say k will be assumed to be algebraically closed and even--to begin with-- of characteristic zero.

It goes without saying that the equivalence relation which we are looking for is one which is much weaker than strict analytical equivalence (i.e., isomorphism of the completions of the local rings of P and P'). Each class of equivalent algebroid singularities will give rise to a variety of biholomorphic moduli (the quotient space of that class, modulo analytic equivalence). One should not expect, however, that the variety of moduli in this context will be irreducible, or even equidimensional. Examples to the contrary can already be given in the case of singularities of algebroid plane curves.

Similarly, if we are in the complex domain, we envisage an equivalence relation which is much stronger than topological equivalence of the two varieties V and V', locally at P and P' respectively. Thus, if V is an algebroid plane curve, the only topological invariant of V is the number of irreducible analytical brancheswhich V has at the point P. However, if we deal only with normal points, then one can expect that in this case the relationship between topological equivalence and algebro-geometric equivalence will be much less casual than in the general case. The only non-trivial result which we have in this connection is Mumford's theorem that if an algebraic surface is topologically a manifold at a normal point then that point is a simple point of the surface.

Still with reference to the complex domain, the set-up in regard to the connection between topological equivalence and the (hypothetical) algebro-geometric equivalence, changes radically if our varieties V and V' are embedded varieties, i.e., are varieties of dimension r, embedded (locally at P and P' respectively) in affine $(r + 1)$-spaces A, A', and if we look at the complementary spaces $A - V$, $A' - V'$. Then one may conjecture that P and P' are equivalent singularities if and only if the spaces $A - V$ and $A' - V'$ are homeomorphic (locally at P and P'). This is only known to be true in the case $r = 2$.

§2. <u>The case of plane algebroid curves.</u>

This classical case, in which everything concerning equivalence is well-known, is nevertheless a very important case, because it contains the germ of all possible generalizations. One must, however, have a second look at this classical case, using a somewhat more sophisticated approach than the one used by Noether and Enriques in their study of the composition of singularities. Above all, one must devise in this case a definition of equivalence which does not presuppose a detailed analysis of the singularity, for in the higher dimensional case such an analysis is a hopeless undertaking. Let me give you three such definitions and say that it can be proved that they are all equivalent (the proofs are not completely trivial) We assume throughout that the ground field k is algebraically closed (of arbitrary characteristic) .

If C is an algebroid plane curve, with origin P we denote by $m_P(C)$ the multiplicity of the point P. If $p_1, p_2, \ldots, p_t$ are the distinct tangents of C at P ($t \leqq m_P(C)$) , then we denote by C_ν ($\nu = 1, 2, \ldots, t$) the union of all irreducible branches of C which are tangent to p_ν, and we call $C_1, C_2, \ldots, C_t$ the <u>tangential components</u> of C .

Let D be another plane algebroid curve, with some origin Q. We <u>assume that</u> C <u>and</u> D <u>have the same number</u> h <u>of irreducible branches,</u>

and we denote by $\delta_1, \delta_2, \ldots, \delta_h$ the irreducible branches of D.

DEFINITION . A (1,1) mapping π of the set of branches $\gamma_1, \gamma_2, \ldots, \gamma_h$ of C onto the set of branches $\delta_1, \delta_2, \ldots, \delta_h$ of D is said to be a tangentially stable pairing $\pi : C \longrightarrow D$ between the branches of C and those of D, if the following condition is satisfied: given any two branches γ_i and γ_j of C, the corresponding branches $\pi(\gamma_i)$ and $\pi(\gamma_j)$ of D have the same tangent if and only if γ_i and γ_j have the same tangent.

Assume that there exists a tangentially stable pairing $\pi : C \longrightarrow D$ between the branches of C and the branches of D. Then it is clear that C and D have the same number t of distinct tangent lines and that π induces a (1,1) mapping of the set $\{p_1, p_2, \ldots, p_t\}$ of tangent lines of C onto the set $\{q_1, q_2, \ldots, q_t\}$ of tangent lines of D. We choose our indexing of these tangent lines in such a way that p_ν and q_ν are paired in this induced mapping, and we denote by C_ν (resp. D_ν) the tangential component of C (resp., D) associated with p_ν (resp., q_ν). Then it is clear that for each $\nu = 1, 2, \ldots, t$, $\pi_\nu : C_\nu \longrightarrow D_\nu$ of the set of branches of C_ν onto the set of branches of D_ν (the pairing π_ν is trivially tangentially stable, since both C_ν and D_ν have only one tangent line).

Assume that there exists a tangentially stable pairing $\pi : C \longrightarrow D$ between the branches of C and the branches of D. Then it is clear that C and D have the same number t of distinct tangent lines and that π induces a $(1,1)$ mapping of the set $\{p_1, p_2, \ldots, p_t\}$ of tangent lines of C onto the set $\{q_1, q_2, \ldots, q_t\}$ of tangent lines of D. We choose our indexing of these tangent lines in such a way that p_ν and q_ν are paired in this induced mapping, and we denote by C_ν (resp. D_ν) the tangential component of C (resp., D) associated with p_ν (resp., q_ν). Then it is clear that for each $\nu = 1, 2, \ldots, t$, π induces a $(1,1)$ mapping $\pi_\nu : C_\nu \longrightarrow D_\nu$ of the set of branches of C_ν onto the set of branches of D_ν (the pairing π_ν is trivially tangentially stable, since both C_ν and D_ν have only one tangent line).

Let π and π_ν be as above (π-tangentially stable), let T be a locally quadratic transformation with center at the origin of P of C and let S be a locally quadratic transformation with center at the origin Q of D. Let $C' = T(C)$, $C'_\nu = T(C_\nu)$, $D' = S(D)$, $D'_\nu = S(D_\nu)$ be the proper transforms. It is clear that π_ν induces a $(1,1)$ mapping π'_ν of the set of branches of C'_ν onto the set of branches of D'_ν. Namely, if we assume that the branches of C and D have been so indexed that $\pi(\gamma_i) = \delta_i$, for $i = 1, 2, \ldots, h$, then we set $\pi'_\nu(\gamma'_i) = \delta'_i$, where $\gamma'_i = T(\gamma_i)$ and $\delta'_i = S(\delta_i)$. The pairing

π'_ν / $C'_\nu \longrightarrow D'_\nu$ between the branches of C'_ν and the branches of D'_ν is, however, not necessarily tangentially stable.

An algebroid curve C is <u>regular</u> if its origin P is a simple point of C, i.e., if $m_P(C) = 1$. If P is a <u>singular</u> point (i.e., if $m_P(C) > 1$), then we can resolve the singularity of C at P by a finite number of locally quadratic transformations. By a sequence of <u>successive quadratic transforms of</u> C we mean a sequence $\{ C, C', C'', \ldots, C^{(i)}, \ldots \}$ of algebroid curves $C^{(i)}$ such that for each i, $C^{(i+1)}$ is a connected component of the proper transform of $C^{(i)}$ under a locally quadratic transformation whose center is the origin of $C^{(i)} (C^{(0)} = C)$. The fact that the singularity of C can be resolved can then be stated as follows : there exists an integer N such that in <u>any</u> sequence of successive quadratic transforms of C, the curves $C^{(i)}$ are regular if $i \geqq N$. We denote by $\sigma(C)$ the smallest integer N with the above property ($\sigma(C) = 0$ if and only if C itself is a regular curve).

It is clear that if $C'_1, C'_2, \ldots, C'_t$ are the connected components of the proper quadratic transform $T(C)$ of C, and if $\sigma(C) > 0$, then $\sigma(C'_\nu) < \sigma(C)$ for $\nu = 1, 2, \ldots, t$. Our first definition of equivalence of algebroid curves proceeds by induction on $\sigma(C)$.

Let π / $C \longrightarrow D$ be a pairing between the branches of C and the branches of D (it is already assumed that C and D have the same number h of branches). If C is regular (whence $\sigma(C) = 0$), then C (and therefore also D) has only one branch, $\pi : C \longrightarrow D$ is uniquely determined, and we say that π is an <u>(a)-equivalence</u> if also D is a regular curve. Assume that for all pairs of algebroid curves Γ, Δ with the same number of branches and such that $\sigma(\Gamma) < \sigma(C)$ it has already been defined what is to be meant by saying that a pairing $\Gamma \longrightarrow \Delta$ between the branches of Γ and the branches of Δ is an (a)-equivalence. Then we define an (a)-equivalence between C and D as follows (we use the notations introduced earlier in this section):

DEFINITION 1. <u>An</u> (a)-<u>equivalence</u> $\pi : C \longrightarrow D$ <u>is a pairing</u> π <u>between the branches of</u> C <u>and the branches of</u> D <u>having the following properties:</u>

1) π <u>is tangentially stable.</u>

2) <u>If</u> $\delta_i = \pi(\gamma_i)$ $(i = 1, 2, \ldots, h)$, <u>then</u> $m_P(\gamma_i) = m_Q(\delta_i)$.

3) <u>The pairing</u> $\pi'_\nu : C'_\nu \longrightarrow D'_\nu$ $(\nu = 1, 2, \ldots, t)$ <u>is an</u> (a)-<u>equivalence.</u>

We now proceed to our second definition of equivalence between algebroid singularities. If T is our quadratic transformation, with center P then T blows up P into the line $x' = 0$ of the (x', y')-plane. We denote this line by $\mathcal{E}'$ and we refer to $\mathcal{E}'$ as the <u>exceptional curve</u> of T. If C_ν is a tangential component of C and $C'_\nu = T(C_\nu)$

is the proper T-transform of C_ν, then $\mathcal{E}'$ contains the origin P'_ν of C'_ν, but $\mathcal{E}'$ is not a component of C'_ν. We denote by C'^*_ν the algebroid curve $C'_\nu \cup \mathcal{E}'$ and we call C'^*_ν the <u>total</u> T-transform of C_ν; in symbols: $C'^*_\nu = T\{C_\nu\}$. We set $C'^* = T(C) \cup \mathcal{E}'$ and we call C'^* the <u>total</u> T-transform of C. Note that $m_{P_\nu}(C'^*_\nu)$ is always ≥ 2.

It is known that after a finite number of successive quadratic transformations one can reach a stage where the <u>total</u> transform of C <u>has only ordinary double points.</u> More precisely: there exists an integer $N \geq 0$ (depending on C) with the following property: if $\{C, C'^*, C''^*, \ldots, C^{(i)*}, \ldots\}$ is <u>any</u> sequence of algebroid curves such that for any i we have $C^{(i+1)*} = C^{(i+1)} \cup \mathcal{E}^{i+1}$, where $C^{(i+1)}$ is a connected component of the proper quadratic transform $T^{(i)}(C^{(i)})$ of $C^{(i)}$($T^{(i)}$ being a quadratic transformation with center at the origin $P^{(i)}$ of $C^{(i)}$) and $\mathcal{E}^{(i+1)}$ is the exceptional curve of $T^{(i)}$, then for $i \geq N$ the origin $P^{(i)}$ of $C^{(i)*}$ is an ordinary double point of $C^{(i)*}$. We denote by $\sigma^*(C)$ the smallest integer N having the above property.

It is clear that $\sigma^*(C) = 0$ if and only if the origin P of C is an ordinary double point of C. If C is a regular curve then a strict interpretation of our definition of $\sigma^*(C)$ would require to set $\sigma^*(C) = 1$. However, we agree to set $\sigma^*(C) = 0$ also if C is a regular curve (this could also have been achieved by a slight change in our general definition of $\sigma^*(C)$). It is easily seen that $\sigma^*(C) = 1$ if and only if P is an ordinary s-fold point of C and $s > 2$.

Let C and D have the same number of branches and let $\pi : C \longrightarrow D$ be a pairing of the branches of C with the branches of D. If $\sigma^*(C) = 0$, i.e., if P is either a simple point or an ordinary double point of C, then we shall say that π is a (b)-equivalence between C and D if and only if also $\sigma^*(D) = 0$, i.e., if and only if the origin Q of D is a simple point or an ordinary double point of D according as P is a simple point or an ordinary double point of C. Assume that for all pairs Γ, Δ of algebroid curves, with the same number of branches, such that $\sigma^*(\Gamma) < \sigma^*(C)$, it has already been defined what is meant by saying that a pairing $\Gamma \longrightarrow \Delta$ between the branches of Γ and the branches of Δ is a (b)-equivalence. Then we define a (b)-equivalence between C and D as follows:

DEFINITION 2. A (b)-equivalence $\pi: C \longrightarrow D$ is a pairing π between the branches of C and the branches of D, having the following properties:

1) π is tangentially stable.

2) The pairings $\pi'_\nu : C'_\nu \longrightarrow D'_\nu$ $(\nu = 1, 2, \ldots, t)$ are (b)-equivalences.

3) If E' and E' are the exceptional curves of the quadratic transformations T and S respectively (having centers at P

<u>and</u> Q), if $C'^{*}_{\nu} = C'^{*}_{\nu} \cup \mathcal{E}'$, $D'^{*}_{\nu} = D'_{\nu} \cup E'$, <u>and if we extend the pairing</u> π'_{ν} <u>to a pairing</u> $\pi'^{*}_{\nu} : C'^{*}_{\nu} \longrightarrow D'^{*}_{\nu}$ <u>by setting</u> $\pi'^{*}_{\nu}(\mathcal{E}') = E'$, <u>then</u> π'^{*}_{ν} <u>is a</u> (b)-<u>equivalence</u>.

Note that conditions 1) and 2) of this definition are identical with the conditions 1) and 3) of Definition 1 ; condition 2) of Definition 1 has been deleted and has been replaced in Definition 2 by condition 3). Thus the equality of the multiplicities of corresponding branches under π is not explicitly postulated in Definition 2.

We now give a third definition of equivalence of algebroid singularities, which we shall refer to as <u>formal equivalence</u>. Again we proceed by induction on $\sigma^{*}(C)$, where we agree that if $\sigma^{*}(C) = 0$ formal equivalence coincides with (b)-equivalence.

DEFINITION 3. <u>Given two algebroid curves</u> C , D ,<u>having the same number of branches, we say that</u> C <u>and</u> D <u>are formally equivalent if there exists a tangentially stable pairing</u> $\pi : C \longrightarrow D$ <u>between the branches of</u> C <u>and the branches of</u> D <u>such that (in our previous notations)</u> :

1) C'_{ν} <u>and</u> D'_{ν} <u>are formally equivalent</u> ($\nu = 1, 2, \ldots, t$)

2) C'^{*}_{ν} <u>and</u> D'^{*}_{ν} <u>are formally equivalent</u> ($\nu = 1, 2, \ldots, t$) .

Note that this definition does not say anything about the nature of the pairings $\pi'_{\nu} : C'_{\nu} \longrightarrow D'_{\nu}$ and $\pi'^{*}_{\nu} : C'^{*}_{\nu} \longrightarrow D'^{*}_{\nu}$ induced by π .

Condition 1) merely requires that there exist, for each $\nu = 1, 2, \ldots, t$, some tangentially stable pairing $\rho'_\nu : C'_\nu \longrightarrow D'_\nu$ satisfying the conditions of the above inductive definition; and similarly, condition 2) requires that there exist a tangentially stable pairing $\rho'^*_\nu : C'^*_\nu \longrightarrow D'^*_\nu$ satisfying similar conditions. It is not even required that ρ'^*_ν be an extension of ρ'_ν. For this reason, Definition 3 is the most subtle (and also the weakest) of our three definitions of equivalence. The fact that these three definitions are all equivalent to each other is therefore not devoid of interest.

REMARK 1. In the case of characteristic $p \neq 0$ the following example poses the question of whether one should not attempt to look for a finer definition of equivalence in that case:

$$C : f = y^p + x^{2p+1} + ax^{2p-1}y = 0, \quad a \neq 0;$$

$$D : g = y^p + x^{2p+1} = 0.$$

It is easily seen that $C \equiv D$ in the sense of the preceding definitions. However, the module of derivations of the local ring of D is free (since $\frac{\partial g}{\partial y} = 0$), while the corresponding module for C is not free. Is such a qualitative difference between the two local rings compatible with a reasonable definition of equivalence?

REMARK 2. It is possible to generalize the definitions 1 and 2 to the case of a pair of arbitrary local rings of dimension 1, despite the fact that the numerical character similar to $\sigma(C)$ is not always available in the abstract case and that therefore the definition cannot be by induction. (It is known that it may not be possible to resolve a non-regular local ring of dimension 1 by successive quadratic transformations)

§3. Analytic families of algebroid curves; equisingularity in codimension 1.

Instead of attempting to establish an equivalence relation between two given singularities, one may try a less static and more fruitful approach, in which one considers an analytic family of singularities:

$$f(\{x\}; \{t\}) = 0, \tag{1}$$

where f is a power series in the coordinates $\{x\} = \{x_1, x_2, \ldots, x_{s+1}\}$ and the parameters $\{t\} = \{t_1, t_2, \ldots, t_\rho\}$, and where we assume that $f(\{0\}; \{t\})$ is identically zero. We have here a ρ-dimensional family of s-dimensional algebroid varieties W_t, embedded in an affine $(s+1)$-space and having a singular point at the origin $\{x\} = 0$. As one considers the specialization $\{t\} \longrightarrow \{0\}$, one may pose the following problem:

<u>Establish criteria which will give a meaning to the statement that the</u>

specialized variety W_0 has the same singularity at the origin as does the general member W_t of the family.

We can interpret equation (1) as defining an $(s + \rho)$ -dimensional embedded variety V, in the affine space of the $s + \rho + 1$ variables x and t. This variety V carries the irreducible subvariety $M: \{x\} = 0$, of dimension ρ (and codimension s). If we denote by P_t the general point $(\{0\}, \{t\})$ of M and by P_0 the special point $(\{0\}, \{0\})$ of M, then W_t is a section of V through P_t, transversal to M, and W_0 is a section of V through P_0, also transversal to M. Furthermore P_0 is a single point of M. One would not be far off the right track were one to say that the singularity of V at the general point P_t of M is the same as the singularity of V at the special point P_0 of M if and only if the transversal sections W_t and W_0 have equivalent singularities at P_t and P_0 respectively. We could therefore tentatively define equisingularity of an embedded variety V, along an irreducible subvariety M of V, at a simple point P_0 of M, as follows:

"DEFINITION "3. V is equisingular along M, at P_0, if there exists a section W_0 of V at P_0, transversal to M, and a section W_t of V at the general point P_t of M, also transversal to M, such that the singularity of W_0 at P_0 is equivalent to the singularity of W_t at P_t.

The trouble with this "definition" is that it is no definition at all, as long as we do not know what we mean by saying that W_0 and W_t have equivalent singularities. However, we can begin by testing this definition in the case in which the codimension s of M is equal to 1, in which case the transversal section are embedded algebroid curves, and for these we know what we mean by equivalent singularities. One obtains in this case a very satisfactory result at least in characteristic zero, via the following:

THEOREM I. *Let* $f(x, y; \{t\}) = 0$ *be an analytic family of plane algebroid curves* C_t, *all containing the origin* $x = y = 0$, *and defined over an algebraically closed ground field* k *of characteristic zero. Let* $C_0 : f(x, y; \{0\}) = 0$ *be the specialization of* C_t *for* $\{t\} \longrightarrow 0$. *Assume that* f *is regular in* y, *and let* $\triangle^y f$ *be the* y-*discriminant of* f ($\triangle^y f \in k[[x,\{t\}]]$). *Write* $\triangle^y f = \mathcal{E}(x, \{t\})x^N$, *where* $N \geqq 0$ *and* $\mathcal{E}(x, t) \in k[[x, \{t\}]]$ *is such that* $\mathcal{E}(0, \{t\}) \neq 0$. *Then the following is true:*

1) *If* $\mathcal{E}(0,\{0\}) \neq 0$, *then* C_t *and* C_0 *are equivalent.*

2) *Conversely, if* $C_t \equiv C_0$ *and if the line* $x = 0$ *is not tangent to* C_0, *then* $\mathcal{E}(0, \{0\}) \neq 0$.

3) *More generally, if* $C_t \equiv C_0$ *and if the line* $x = 0$ *has the same intersection multiplicity with* C_t *and* C_0, *then* $\mathcal{E}(0, \{0\}) \neq 0$.

We now interpret this theorem by looking at $f = 0$ as the equation of an algebroid embedded variety V of dimension $r = \rho + 1$, where ρ is, as above, the number of parameters t_i. The r elements $x, t_1, t_2, \ldots, t_\rho$ are parameters of the local ring of V at the point P_0. The equation $\triangle^y f = 0$, i.e., $\varepsilon(x, \{t\})x^N = 0$, is an equation of the critical variety $\triangle$ of the projection of V onto the affine space of the variables $x, t_1, t_2, \ldots, t_\rho$. To say that $\varepsilon(0, \{0\}) \neq 0$ means to say that $\triangle$ is the non-singular hypersurface $x = 0$ in that space. Note that $\triangle$ is then the projection of our subvariety M of V, of codimension 1, defined by $x = y = 0$. One then deduces from Theorem 1 the following:

THEOREM 2. If cod $M = 1$, and M is part of the singular locus of V, then V is equisingular along M at P_0, if and only if there exist local parameters $x_1, x_2, \ldots, x_r$ of V at P_0, such that the critical variety $\triangle$ of the projection π of V onto the space of these parameters has a simple point at $P_0' = \pi(P_0)$. Furthermore, if V is equisingular along M at P_0, and $x_1, x_2, \ldots, x_r$ are arbitrary transversal local parameters (by this we mean that the line $x_1 = x_2 = \ldots = x_r = 0$ is not tangent to V at P_0), then the corresponding ciritical variety $\triangle$ has necessarily a simple point at P_0'.

At the Scientific conference at Yeshiva University last October, I spoke extensively about equisingularity in the case of codimension $s = 1$ and gave a number of other criteria of equisingularity in this case (always for characteristic zero). A limited number of copies of my Yeshiva lecture will be made available later on in informal discussions for those who are interested.

§4. Testing a general definition of equisingularity in a special case.

We maintain the assumption that the ground field k is algebraically closed and of characteristic zero. We consider again an r-dimensional algebroid variety V, embedded in an affine $(r+1)$-space, an irreducible singular subvariety M of V, of codimension s on V, and a simple point P_0 of M. We shall define equisingularity of V at P_0, along M, by induction on s. If $x_1, x_2, \ldots, x_r$ are parameters of the local ring of V at P_0, we consider the projection π of V onto the affine r-space of the x_i, and we denote by Δ_x the corresponding critical hypersurface in that space. Then $\pi(M) \subset \Delta_x$, and $\pi(M)$ has codimension $s - 1$ on Δ_x.

DEFINITION 3. (Conjectural). V <u>is equisingular at the point</u> P_0 <u>along</u> M, <u>if there exist parameters</u> $x_1, \ldots, x_r$ <u>such that</u> Δ_x <u>is equisingular at the point</u> $\pi(P_0)$, <u>along</u> $\pi(M)$.

I have no general theory of equisingularity, based on this inductive definition. I will discuss this definition in a special, but theoretically important case.

There is one obvious and uncontestable case of equisingularity. That is the case in which V, as an algebroid variety, is locally, at P_0, a direct (analytic) product of M and a transversal section W_0, at P_0. That means that, for a suitable choice of the coordinates $x_1, x_2, \ldots, x_{r+1}$ in the ambient affine space of V, the equation of V involves only $s + 1$ of the coordinates x_i, say

$$V: \ f(x_1, x_2, \ldots, x_{s+1}) = 0,$$

and M is the subvariety $x_1 = x_2 = \ldots = x_{s+1} = 0$. The transversal section W_0 at P_0 (the origin $x_1 = x_2 = \ldots = x_{r+1} = 0$) is given by the same equation $f = 0$, in the space of the $s + 1$ coordinates $x_1, x_2, \ldots, x_{s+1}$. If σ and $\mathcal{O}$ denote respectively the local ring of V and W_0 at P_0, and if, for the sake of clarity, we denote the remaining coordinates $x_{s+2}, x_{s+3}, \ldots, x_{r+1}$ by $t_1, t_2, \ldots, t_\rho$ ($\rho = r - s$), then

$$\mathcal{O} = \sigma[[t_1, t_2, \ldots, t_\rho]],$$

and $t_1, t_2, \ldots, t_\rho$ are analytically independent over σ. We say in this case that V is <u>analytically equisingular at</u> P_0, <u>along</u> M.

Now, let us assume that the <u>critical variety</u> Δ_x (in Definition 3) <u>is analytically equisingular at the point</u> $\pi(P_0)$, <u>along the variety</u> $\pi(M)$.

Our variety V is then given by an equation

$$V: f(x_1, x_2, \ldots, x_s ; t_1, t_2, \ldots, t_\rho ; y) = 0 \quad (\rho = r - s),$$

where the power series f is regular in y; the variety M is defined by $x_1 = x_2 = \ldots = x_s = y = 0$; and our assumption is that the y-discriminant $\Delta^y f$ of f is of the form

$$(2) \qquad \Delta^y f = \varepsilon(\{x\}, \{t\}) \cdot D(x_1, x_2, \ldots, x_s), \quad \varepsilon(\{0\}, \{0\}) \neq 0 .$$

Note that in the case of equisingularity in codimension $s = 1$, this assumption is automatically satisfied, for in that case we have, by Theorem 2: $\Delta^y f = \varepsilon(\{x\}, \{t\}) x_1^N$.

Another, theoretically important case in which this assumption is satisfied is the one in which the critical variety Δ_x has along $\pi(M)$ a normal crossing. Necessarily, we will have along $\pi(M)$ a normal crossing of s regular hypersurfaces, since $\operatorname{cod}_{\Delta_x} \pi(M) = s - 1$. That means that $\Delta^y f$ will be of the form (2), with $D = x_1^{N_1} x_2^{N_2} \ldots x_s^{N_s}$, $N_j \geq 1$.

Under the above assumption (2), the following algebraic facts can be established:

Let $\mathcal{O}$ be the local ring of V at the point P_0 (the origin $\{x\} = \{t\} = y = 0$), and let σ be the local ring of the transversal section $W_0: t_1 = t_2 = \ldots = t_\rho = 0$ at the same point P_0. Thus

(3) $$\mathfrak{O} = k[[\{x\},\{t\}]][y] = k[[\{x\},\{t\},Y]]/(f(\{x\},\{t\},Y)\ ,$$

and

(4) $$\mathfrak{o} = k[[\{x\}]][\eta] = k[[\{x\},Y]]/(f_0(\{x\},Y)\ ,$$

where

$$f_0(\{x\},Y) = f(\{x\},\{0\},Y)\ .$$

Let $\mathfrak{O}'$ and $\mathfrak{o}'$ be the integral closure of $\mathfrak{O}$ and $\mathfrak{o}$ respectively (in the total rings of quotients of these two local rings). Then

(a) __There is a natural injection of__ $\mathfrak{o}'$ __into__ $\mathfrak{O}'$, __and__ (__after identification__ $\mathfrak{o}' \subset \mathfrak{O}'$) __it is true that the elements__ $t_1, t_2, \ldots, t_p$ __of__ $\mathfrak{O}'$ __are analytically independent over__ $\mathfrak{o}'$, __and__ $\mathfrak{O}'$ __is the power series ring__ $\mathfrak{o}'[[t_1, t_2, \ldots, t_p]]$.

By (a) we have for y a power series expansion of the form

(5) $$y = \eta + \sum u_i t_i + \sum u_{ij}\ t_i\ t_j + \ldots\ ,\quad (u_i, u_{ij}, \ldots \in \mathfrak{o}')$$

where η is the element which occurs in (4). Assumption (2) imposes, however, additional conditions on the coefficients $u_i, u_{ij}, \ldots$ of the power series (5). We shall now state these conditions.

Since we have assumed that f_0 has no multiple factors, the total quotient ring K' of $\mathfrak{o}$ (= total quotient ring of $\mathfrak{o}'$) is a direct sum of fields $K'_i = K'e_i$ (say, $i = 1, 2, \ldots, h$), where $1 = e_1 + e_2 + \ldots + e_h$ is the decomposition of 1 into mutually orthogonal idempotents. Each field K'_i is an algebraic extension of the field Ke_i, where $K = k\{\{x\}\}$. We consider a fixed splitting field F' of the y-polynomial f_0, over K, and we embed each K'_i in F' by an isomorphism $K'_i \longrightarrow F_i$ which is an extension of the natural isomorphism $Ke_i \longrightarrow K$. Let $F'_i \supset F_i$ be the least Galois extension of K which contains F_i. Then F' is the composition of the h fields F'_i (the F'_i are splitting fields of the h irreducible factors of f_0).

Now, let ξ be any element of K'. For each $i = 1, 2, \ldots, h$, we denote by $\xi_i^{(1)}, \xi_i^{(2)}, \ldots, \xi_i^{(n_i)}$ the conjugates, over K of the element of F_i which corresponds to ξe_i in the above embedding $K'_i \longrightarrow F_i$ of K'_i in F; here n_i is the relative degree of F_i over K. Let R be the set of elements ξ of $\mathcal{O}'$ which have the following property: <u>For any</u> $i, j = 1, 2, \ldots, h$ <u>and any</u> $\alpha = 1, 2, \ldots, n_i$, $\beta = 1, 2, \ldots, n_j$, <u>the quotients</u>

$$(6) \qquad (\xi_i^{(\alpha)} - \xi_j^{(\beta)})/(\eta_i^{(\alpha)} - \eta_j^{(\beta)})$$

<u>are integral over</u> $k[[\{x\}]]$.

It is easily seen that R is a ring between $\mathcal{O} = k[[\{x\}]][\eta]$ and $\mathcal{O}'$.

Then we have

(b) A necessary and sufficient condition that the discriminant $\triangle^{y}f$ be of the form (2) is that the coefficients $u_i, u_{ij}, \ldots$ of the power series (5) belong to R .

If our variety V was normal at P_0, then $\mathcal{O} = \mathcal{O}'$, $\sigma = \sigma'$, $\mathcal{O} = \sigma[[t_1, t_2, \ldots, t_p]]$, and we have in this case the trivial situation of analytical equisingularity of V along M, at P_0. But if V is not normal, then R will be in general a proper overring of σ, and if we choose the coefficients $u_i, u_{ij}, \ldots$ in R , but not all in σ , then we get a situation of equisingularity which is not analytical. Thus this procedure gives us an effective tool for a general construction of an equisingularity phenomenon of the non-trivial (i.e., non analytic) type.

If we are in the complex domain then (6) and the fact that the coefficient of the power series (2) are all in R shows that

$$(7) \qquad \lim_{\{t\} \to 0} \frac{y^{(\alpha)} - y^{(\beta)}}{\eta^{(\alpha)} - \eta^{(\beta)}} = 1 ,$$

where the $y^{(\alpha)}$ are the roots of f and the $\eta^{(\alpha)}$ are the roots of $f^{(0)}$ ($\alpha, \beta = 1, 2, \ldots, n = n_1 + n_2 + \ldots + n_h$, $\alpha \neq \beta$).
By means of (7) it is possible to extend a proof given by Whitney in the case of codimension 1 and show that in the ambient affine $(r+1)$-space

of V, the variety V can be isotopically deformed into the direct product of W_0 and M. This constitutes a fairly conclusive text of the correctness of the inductive definition 3 of equisingularity in this particular case.

On the structure of compact complex analytic surfaces

by K. Kodaira

By a <u>surface</u> we shall mean a compact complex manifold of complex dimension 2. We fix our notation as follows.

S : a surface

b_ν : the ν - th Betti number of S,

c_ν : the ν - th Chern class of S,

$\mathcal{O}$: the sheaf over S of germs of holomorphic functions,

$q = \dim H^1(S, \mathcal{O})$: the irregularity of S,

$P_g = \dim H^2(S, \mathcal{O})$: the geometric genus of S.

Note that c_1^2 and c_2 are (rational) integers.

By a theorem of Grauert [2], any surface is obtained from a surface containing no exceptional curve (of the first kind) by means of a finite number of quadric transformations. Hence, in order to study the structure of surfaces, it suffices to consider surfaces containing no exceptional curves. In what follows we assume that all surfaces under consideration contain no exceptional curves.

DEFINITION 1. By an elliptic surface we shall mean a surface S with a holomorphic map ϕ of S onto a non-singular algebraic curve Δ such that the inverse image $\phi^{-1}(u)$ of any general point

$u \in \Delta$ is an elliptic curve. We call Δ the base curve of the elliptic surface S.

DEFINITION 2. (A. Weil). We call a surface S a K3 surface if S is a deformation of a non-singular quartic surface in a projective 3-space.

MAIN THEOREM. Surfaces (containing no exeptional curves) can be classified into the following seven classes :

I) the class of algebraic surfaces with $p_g = 0$;

II) the class of K3 surfaces;

III) the class of complex tori (of complex dimension 2) ;

IV) the class of elliptic surfaces with $b_1 \equiv 0(2)$, $p_g \geqq 1$, $c_1 \neq 0$;

V) the class of algebraic surfaces with $p_g \geqq 1$, $c_1^2 > 0$;

VI) the class of elliptic surfaces with $b_1 \equiv 0(1)$, $p_g \geqq 1$;

VII) the class of surfaces with $b_1 = q = 1$, $p_g = 0$.

class	b_1	p_g	c_1	c_1^2	structure
I	even	0			algebraic
II	0	1	=0	0	K3 surfaces
III	4	1	=0	0	complex tori
IV	even	+	$\neq 0$	0	elliptic
V	even	+		+	algebraic
VI	odd	+		0	elliptic
VII	1	0			?

An elliptic surface is a deformation of an algebraic surface if and only if its first Betti number is even (see [4]) . Therefore the following theorem follows from the main theorem.

THEOREM . A surface is a deformation of an algebraic surface if and only if its first Betti number is even.

Remark: The class VII contains many elliptic surfaces. In fact, for any preassigned finite abelian group A , we find an elliptic surface of the class VII whose first torsion group is isomorphic to A . We obtain examples of non-elliptic surfaces of the class VII as follows : Let $\mathbb{C}^2$ denote the space of two complex variables (z_1, z_2) and let $U = \mathbb{C}^2 - (0,0)$. Choose a properly discontinuous group $\mathcal{G}$ of analytic automorphisms without fixed points of U in an appropriate manner. Then the quotient surface $S = U/\mathcal{G}$ is a non-elliptic surface of the class VII. Note that $S = U/\mathcal{G}$ is a deformation of an elliptic surface. As far as we know there is no example of a surface which cannot be deformed into surfaces with non-constant meromorphic functions.

We shall outline a proof of the main theorem. Let $\mathcal{O}^*$ be the multiplicative sheaf over S of germs of non-vanishing holomorphic functions and let $\mathbb{Z}$ denote the ring of rational integers. We have the exact sequence

$$(1) \qquad \cdots \to H^1(S,\mathcal{O}) \to H^1(S,\mathcal{O}^*) \xrightarrow{\delta^*} H^2(S,\mathbb{Z}) \to H^2(S,\mathcal{O}) \to \cdots$$

Each element F of $H^1(S, \mathcal{O}^*)$ represents a complex line bundle over S and $c(F) = \delta^* F$ is the Chern class of F. Let $\mathcal{O}(F)$ denote the sheaf over S of germs of holomorphic sections of F. In the case of complex line bundles over surfaces, the Riemann-Roch-Hirzebruch theorem can be formulated as follows :

$$\sum_{\nu=0}^{2} (-1)^\nu \dim H^\nu(S, \mathcal{O}(F)) = \frac{1}{2}(c^2 + c_1 c) + \frac{1}{12}(c_1^2 + c_2), \qquad c = c(F) \tag{2}$$

(see Atiyah and Singer [1]). This theorem implies the Noether formula

$$12(p_g - q + 1) = c_1^2 + c_2 \tag{3}$$

and the Riemann-Roch inequality

$$\dim H^0(S, \mathcal{O}(F)) + \dim H^0(S, \mathcal{O}(K-F)) \geq \frac{1}{2}(c^2 + c_1 c) + p_g - q + 1 , \tag{4}$$

where K denotes the canonical bundle of S.

THEOREM 1. Every holomorphic 1-form on a surface is d-closed.

THEOREM 2. Let $\varphi_1, \varphi_2, \ldots, \varphi_n$ be holomorphic 1-forms on S. If $\varphi_1, \ldots, \varphi_n$ are linearly independent, then the d-closed 1-forms $\varphi_1, \ldots, \varphi_n, \bar{\varphi}_1, \ldots, \bar{\varphi}_n$ are d-cohomologically independent.

Letting $\{ \Gamma_1, \ldots, \Gamma_j, \ldots, \Gamma_{b_2} \}$ be a Betti base of 2-cycles on S and denoting by $I(\Gamma_j, \Gamma_k)$ the intersection multiplicity of Γ_j

and Γ_k, we define b^+ and b^- to be respectively the number of positive and negative eigenvalues of the non-singular symmetric matrix $(I(\Gamma_j, \Gamma_k))$. Moreover we denote by h the number of linearly independent holomorphic 1-forms on S. With the aid of Theorems 1 and 2, we obtain from the Hirzebruch index theorem and the Noether formula (3) the equality

$$2q - b_1 + b^+ - 2p_g = 1 ,$$

while we have the inequalities

$$q \geqq \frac{1}{2} b_1 \geqq h \geqq b_1 - q , \qquad b^+ \geqq 2p_g .$$

Hence we obtain the following

THEOREM 3. If b_1 is even, then $b_1 = 2q$, $b^+ = 2p_g + 1$ and $h = q$. If b_1 is odd, then $b_1 = 2q - 1$, $b^+ = 2p_g$ and $h = q - 1$.

COROLLARY. We have the formula

$$(5) \qquad c_1^2 + 8q + b^- = \begin{cases} 10p_g + 9 , & \text{if } b_1 \text{ is even} , \\ 10p_g + 8 , & \text{if } b_1 \text{ is odd} . \end{cases}$$

Let us consider the case in which b_1 is even. By the above results there exist q linearly independent d-closed holomorphic

1-forms $\varphi_1, \varphi_2, \ldots, \varphi_q$ on S. Let $\{\gamma_1, \ldots, \gamma_j, \ldots, \gamma_{2q}\}$ be a Betti base of 1-cycles on S and let

$$\omega_{\nu j} = \int_{\gamma_j} \varphi_\nu \quad .$$

Then, by Theorem 2, the vectors

$$\omega_j = (\omega_{1j}, \ldots, \omega_{\nu j}, \ldots, \omega_{qj}), \qquad j = 1, 2, \ldots, 2q,$$

are linearly independent with respect to real coefficients and generate a discontinuous subgroup $\mathcal{D}$ of the vector group $\mathbb{C}^q$ of dimension q. We call $\mathcal{A} = \mathbb{C}^q/\mathcal{D}$ the Albanese variety attached to S and define a holomorphic map Φ of S into $\mathcal{A}$ in an obvious manner.

THEOREM 4. If there exist on S two algebraically independent meromorphic functions, then S is an algebraic surface. If there exists on S one and only one algebraically independent meromorphic function, then S is an elliptic surface (see [3]).

The following three theorems follow immediately from this theorem.

THEOREM 5. If there exists on S a complex line bundle F such that $\dim H^0(S, \mathcal{O}(F)) \geqq 2$, then S is either an algebraic surface or an elliptic surface.

THEOREM 6. If $p_g \geqq 2$, then S is either an algebraic surface or an elliptic surface.

THEOREM 7. If $h \geqq 3$, then S is either an algebraic surface or an elliptic surface.

Combining the Riemann-Roch inequality (4) with Theorem 5, we obtain the following two theorems.

THEOREM 8. If there exists on S a complex line bundle F with $c(F)^2 > 0$, then S is an algebraic surface.

THEOREM 9. If $c_1^2 > 0$, then S is an algebraic surface.

THEOREM 10. If b_1 is even and if $p_g = 0$, then S is an algebraic surface.

Proof: Since, by Theorem 3, $b^+ = 1$, there exists an element $c \in H^2(S, \mathbf{Z})$ with $c^2 > 0$. Moreover, since $H^2(S, \mathcal{O})$ vanishes, the exact sequence (1) shows the existence of a complex line bundle F over S with $c(F) = c$. Hence, by Theorem 8, S is an algebraic surface.

LEMMA 1. If $p_g \geqq 1$, then $c_1^2 \geqq 0$.

THEOREM 11. Assume that there exists on S no meromorphic function except constants (and that S contains no exceptional curve). Then the irregularity q of S is not greater than 2. If $q = 2$, then S is a complex torus. If $q = 1$, then the first Betti number b_1 of S is equal to 1 and the geometric genus p_g of S vanishes. If $q = 0$, then the first Chern class c_1 of S vanishes.

Proof: A) The case in which b_1 is even. It follows from Theorems 3, 6, 7 and 10 that $b_1 = 2q$, $q = h \leqq 2$ and $p_g = 1$. Hence, by Lemma 1 and Theorem 9, $c_1^2 = 0$.

i) q is equal to either 2 or 0. In fact, if q were equal to 1, then the Albanese variety $\mathcal{A}$ would be an elliptic curve and the meromorphic functions on $\mathcal{A}$ would induce non-constant meromorphic functions on S.

ii) If $q = 2$, then the Albanese variety $\mathcal{A}$ is a complex torus and Φ maps S biholomorphically onto $\mathcal{A}$.

iii) If $q = 0$, then we have

$$\dim H^0(S, \mathcal{O}(-K)) + \dim H^0(S, \mathcal{O}(2K)) \geqq 2 .$$

Hence, in view of Theorem 5, $\dim H^0(S, \mathcal{O}(-K)) = 1$, while $\dim H^0(S, \mathcal{O}(K)) = p_g = 1$. Consequently K is trivial and c_1 vanishes.

B) The case in which b_1 is odd. It follows from Theorems 3, 6, 7 and 9 that $b_1 = 2q - 1$, $q = h + 1$, $b^+ = 2p_g$, $h \leqq 2$, $p_g \leqq 1$ and $c_1^2 \leq 0$.

i) Suppose that $h = 2$. Then there exist on S two linearly independent holomorphic 1-forms φ_1 and φ_2 and $\varphi_1 \wedge \varphi_2$ does not vanish identically. Hence $p_g = 1$ and, by Lemma 1, $c_1^2 = 0$. The formula (5) then proves that $b^- = -6$. This is a contradiction.

ii) Suppose that $h = 1$. We take a d-closed holomorphic 1-form φ on S and find a 1-form σ of type $(1,0)$ on S such that $d\sigma = \varphi \wedge \bar{\varphi}$ and such that $\sigma + \bar{\sigma}$, φ and $\bar{\varphi}$ generate the d-cohomology group of 1-forms on S. We then obtain multi-valued holomorphic functions w_1 and w_2 on S such that

$$dw_1 = \varphi, \quad dw_2 = \sigma + \bar{w}_1 \varphi.$$

The exterior product $dw_1 \wedge dw_2$ does not vanish at each point of S. Hence the space $\mathbb{C}^2$ of the complex variables w_1 and w_2 forms the universal covering surface of S. The covering transformation group of $\mathbb{C}^2$ over S is generated by the affine transformations

$$g_j : w_1 \longrightarrow w_1 + \alpha_j, \quad w_2 \longrightarrow w_2 + \bar{\alpha}_j w_1 + \beta_j, \quad j = 1, 2, 3, 4,$$

of which the coefficients satisfy the conditions that

$$\alpha_4 = 0, \quad \bar{\alpha}_j \alpha_k - \bar{\alpha}_k \alpha_j = n_{jk} \beta_4, \qquad \text{for } j, k = 1, 2, 3,$$

where the n_{jk} are integers and $n_{23}\beta_4 \neq 0$. It follows that S is an elliptic surface. This contradicts the non-existence of meromorphic functions on S.

iii) Thus we see that $h = 0$ and $q = b_1 = 1$. Therefore the Picard variety $\mathcal{P} = H^1(S,\mathcal{O})/H^1(S, \mathbb{Z})$ is isomorphic to the Lie group $\mathbb{C}/\mathbb{Z}$. Suppose that $p_g = 1$. Then, for <u>each</u> complex line bundle $F \in \mathcal{P}$, the inequality

$$\dim H^0(S,\mathcal{O}(F)) + \dim H^0(S,\mathcal{O}(K-F)) \geq 1$$

holds. It follows that there exist infinitely many irreducible curves on S. This contradicts the non-existence of meromorphic function on S (see [3]), q. e. d.

THEOREM 12. If the irregularity q and the first Chern class c_1 of S both vanish, then S is a $K3$ surface.

<u>Proof:</u> Denoting by Θ the sheaf over S of germs of holomorphic vector fields, we have

$$\dim H^1(S, \Theta) = 20, \quad \dim H^2(S, \Theta) = 0.$$

Hence there exists a complete complex analytic family of small deformations S_t of S depending on 20 effective parameters $(t_1, t_2, \ldots, t_{20})$ (see Kodaira, Nirenberg and Spencer [5]). We find

t such that S_t is a non-algebraic elliptic surface of which the singular fibres are either of type I_1, or of type II (compare [3]). S_t is a fibre preserving deformation of an algebraic elliptic surface B which possesses a global holomorphic section. B can be described explicitly as follows: Let $\mathbb{P}^2$ denote a projective plane on which a system of homogeneous coordinates (x, y, z) is fixed. Take two copies $\mathbb{P}^2 \times \mathbb{C}_0$ and $\mathbb{P}^2 \times \mathbb{C}_1$ of $\mathbb{P}^2 \times \mathbb{C}$ and form their union

$$W = \mathbb{P}^2 \times \mathbb{C}_0 \cup \mathbb{P}^2 \times \mathbb{C}_1$$

by identifying $(x, y, z, u) \in \mathbb{P}^2 \times \mathbb{C}_0$ with $(x_1, y_1, z_1, u_1) \in \mathbb{P}^2 \times \mathbb{C}_1$ if and only if $uu_1 = 1$, $u^4 x_1 = x$, $u^6 y_1 = y$, $z_1 = z$. Then B is the subvariety of W defined by an equation of the form

$$y^2 z - 4x^3 + \tau_0 x z^2 \prod_{\nu=1}^{8} (u - \tau_\nu) + z^3 \prod_{\nu=1}^{12} (u - \sigma_\nu) = 0 .$$

To make explicit the dependence of B on the coefficients $\tau = (\tau_0, \tau_1, \ldots, \tau_8, \sigma_1, \ldots, \sigma_{12})$, we write B_τ for B. Clearly B_τ is a deformation of $B^0 = B_{(1,\ldots,1,0,\ldots,0)}$. Hence S is a deformation of B^0. Let Q denote a non-singular quartic surface in a projective 3-space. The irregularity and the first Chern class of Q both vanish. Hence, by the above result, Q is a deformation of B^0 and, consequently, S is a deformation of Q. Thus we see that S is a K3 surface.

THEOREM 13. If the canonical bundle K of S is trivial, then S is a $K3$ surface, a complex torus, or an elliptic surface of the form $\mathbb{C}^2/G$, where G is a properly discontinuous group of affine transformations without fixed points of the space $\mathbb{C}^2$ of two complex variables z_1, z_2 which leave invariant the 2-form $dz_1 \wedge dz_2$. The first Betti number of the elliptic surface $\mathbb{C}^2/G$ is equal to 3.

LEMMA 2. If b_1 is even, $p_g > 0$ and $c_1 = 0$, then the canonical bundle K of S is trivial.

LEMMA 3. If p_g is positive, $c_1^2 = 0$ and $c_1 \neq 0$, then S is an elliptic surface.

Now, with the aid of Lemmas 1, 2, 3, we derive readily from Theorems 9-13 the main theorem.

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ON DEFORMATIONS AND VARIETIES OF MODULI

T. Matsusaka

§1. The notion of polarization is well-known by now. But we shall start with this definition. Let V be a complete non-singular algebraic variety and $G_a(V)$ the group of V-divisors which are algebraically equivalent to zero. Denote by $T(V)$ the group of torsion divisors on V and by $t(V)$ the order of $T(V)$. We consider a set $\mathfrak{X}$ of V-divisors which is defined by the following conditions:

(a) $\mathfrak{X}$ contains a divisor x on X which is non-degenerate (ample in the sense of Grothendieck);

(b) A V-divisor Y is in $\mathfrak{X}$ if and only if there is a pair (r, s) of integers which are relatively prime to the characteristic p and to $t(V)$ such that $rX \equiv sY \mod G_a(X)$.

We consider that the set $\mathfrak{X}$ defines a structure on V. V, together with this additional structure, is denoted by $\underline{V}$ and is called a <u>polarized variety</u>. We call V the <u>underlying variety</u> of $\underline{V}$ and $\mathfrak{X}$ the <u>structure set</u> of $\underline{V}$. A divisor in $\mathfrak{X}$ is called a <u>polar divisor</u> of $\underline{V}$.

REMARK. If one wants to deal with a variety over a discrete valuation ring, of which V is a generic fibre, it is convenient to take p to be the characteristic of the residue field.

Basically, we shall follow the terminology and conventions of Weil's "Foundations of Algebraic Geometry".

PROPOSITION 1 Let $\underline{V}$ be a polarized variety. Then there is a polar divisor X_o with the following properties: (i) A V-divisor Y is a polar divisor of $\underline{V}$ if and only if it is algebraically equivalent to mX_o where m is an integer; (ii) A non-degenerate polar divisor on $\underline{V}$ is algebraically equivalent to mX_o where m is a positive integer. Moreover, the class of X_o mod $G_a(V)$ is uniquely determined by these conditions.

X_o is called a basic polar divisor of $\underline{V}$. The self-intersection number of X_o is called the rank or the degree of $\underline{V}$. A non-singular subvariety of a projective space can have a natural polarization such that a hyperplane section is a polar divisor. We call it a natural polarization and all such varieties shall be assumed to carry their natural polarization.

Let U be a complete (proper) abstract variety over a discrete valuation ring $\mathfrak{O}$, α the canonical morphism of U onto $\mathfrak{O}$ and $\mathfrak{p}$ the maximal ideal of $\mathfrak{O}$. Then $\alpha^{-1}(\mathfrak{p})$ is called the specialization of $\alpha^{-1}(0)$ over $\mathfrak{O}$. If X is a cycle on a generic fibre $\alpha^{-1}(0)$, rational over the quotient field k of $\mathfrak{O}$, it defines a U-cycle $\tilde{X}$ uniquely such that $\tilde{X} \cdot \alpha^{-1}(0) = X$, and that every component of $\tilde{X} \cap \alpha^{-1}(\mathfrak{p})$ which is simple on U is proper. Then $\tilde{X} \cdot \alpha^{-1}(\mathfrak{p})$ is called the specilization of X over $\mathfrak{O}$. $\tilde{X} \cdot \alpha^{-1}(\mathfrak{p})$ is still the specilization of X over a discrete valuation ring which dominates $\mathfrak{O}$.

Let $\underline{V}$ be a polarized variety, V its underlying variety and k a field of definition of $\underline{V}$. Let $\mathfrak{O}$ be a discrete valuation ring of k, U the variety over $\mathfrak{O}$ with the canonical morphism α whose generic fibre is V and W the special fibre $\alpha^{-1}(\mathfrak{p})$ of U. Assume that W is the underlying variety of a polarized variety $\underline{W}$ and that a basic polar divisor X_0 of $\underline{V}$ specializes to a polar divisor of $\underline{W}$ over $\mathfrak{o}$. Then we say that $\underline{W}$ is the <u>specialization</u> of $\underline{V}$ over $\mathfrak{o}$ and write $\underline{V} \longrightarrow \underline{W}$ ref. $\mathfrak{o}$.

REMARK. When $\underline{V} \longrightarrow \underline{W}$ ref. $\mathfrak{o}$, then rank $(\underline{V}) \geq$ rank $(\underline{W})$. When $\mathfrak{o}$ contains the rational number field, we have rank $(\underline{V})$ = rank $(\underline{W})$ because of Hodge's theorem. On the other hand, when the quotient field of $\mathfrak{o}$ is of characteristic p, Nishi constructed an example such that rank $(\underline{V})$ > rank $(\underline{W})$. In his example, $\underline{W}$ is a suitably polarized Abelian variety of dimension 2 and one can choose $\underline{V}$ so that rank $(\underline{V})$ exceeds a given positive integer.

In general, the concept of specialization is not invariantly attached to isomorphism classes of polarized varieties. However we have the following result.

PROPOSITION 2. Let $\underline{V}$, $\underline{V}'$, $\underline{W}$, $\underline{W}'$ be four polarized varieties, k a common field of definition of $\underline{V}$ and $\underline{V}'$, $\mathfrak{o}$ a discrete valuation ring of k and assume that $(\underline{V}, \underline{V}') \longrightarrow (\underline{W}, \underline{W}')$ ref. $\mathfrak{o}$. When there is an isomorphism f of $\underline{V}$ to $\underline{V}'$ defined over k and when $\underline{W}'$ is not ruled, the graph of f specializes to the graph of an isomorphism between $\underline{W}$ and $\underline{W}'$ over $\mathfrak{o}$.

Let now $\underline{V}$ and $\underline{W}$ be two polarized varieties and the $\underline{V}_i$, for $0 \leq i \leq$ $\underline{V}_0 = \underline{V}$, , $\underline{V}_m = \underline{W}$, be a finite set of polarized varieties. Assume that either $\underline{V}_i$ and $\underline{V}_{i+1}$ are isomorphic to each other, or the one is a specialization of the other over some discrete valuation ring. Then we say that $\underline{W}$ is a deformation of $\underline{V}$, $\underline{V}$ is a deformation of $\underline{W}$ or $\underline{V}$ and $\underline{W}$ are deformations of each other. If rank $(\underline{V}_i) \leq d$ for all i, we say that the deformation is of type d. Denote by Σ the set of deformations of a given polarized variety $\underline{V}$. Denote also by Σ_d the set of deformations of $\underline{V}$ of type d for $d \geq$ rank $(\underline{V})$. In the case of characteristic 0, we have $\Sigma = \Sigma_d$. Therefore, we shall consider only Σ_d from now on. We introduce an equivalence relation $\sim$ in Σ_d. We say that $\underline{U}$ and $\underline{W}$ in Σ_d are equivalent and write $\underline{U} \sim \underline{W}$ if and only if $\underline{U}$ and $\underline{W}$ are isomorphic. The quotient space of Σ_d by this equivalence relation will be called the space of moduli and denoted by $\mathfrak{M}$. Our basic problem is to find out the structure of this space. One is tempted to say that it is an algebraic variety, or at least a finite union of such varieties. Furthermore, one is tempted to say that the largest dimension of the maximal component can be described in terms of numerical invariants of $\underline{V}$ (cf. works of Kodaira-Spencer). As it is well-known, these are true when one deals with curves or polarized Abelain varieties (cf. works of Baily, Mumford). In general Σ_d can be expressed as a union of countably many irreducible algebraic families of polarized varieties up to isomorphisms In order to pursue our problems further, we introduce the concept of a universal family.

Let $\mathcal{F}$ be an algebraic family, i. e. a union of a finite set of irreducible algebraic families, of non-singular varieties in a projective space such that

(i) A member of $\mathcal{F}$ is a member of Σ_d ;

(ii) A member of Σ_d is isomorphic to a member of $\mathcal{F}$.

Then we say that $\mathcal{F}$ is a <u>universal family</u> of Σ_d. The universal family of Σ_d exists if and only if the following is true.

<u>There is a constant</u> c, <u>depending only on</u> Σ_d, <u>such that whenever</u> <u>W</u> <u>is a member of</u> Σ_d <u>and</u> Y <u>a basic polar divisor of</u> <u>W</u>, mY <u>is ample</u> (very ample in the sense of Grothendieck) <u>for</u> $m > c$.

Of course, this conjecture is true when <u>V</u> is a curve or a polarized Abelian variety. When <u>V</u> is a polarized surface, this conjecture is affirmative and can be deduced from the following theorem.

THEOREM 1. Let V be a non-singular complete surface and X a non-degenerate V-divisor. Let $|p_a(X)| < c_1$, $X^{(2)} < c_2$ and $|p_a(V)| < c_3$. Then, there is a constant, depending only on c_1, c_2, c_3, such that mX is ample for $m > c$.

When the dimension of <u>V</u> is higher than 2, nothing is known in general. Kim showed that if <u>V</u> can be mapped into Albanese variety without fundamental subvarieties, the subset of Σ_d consisting of polarized varieties with the same property has a universal family. Assuming only that $\underline{V}$ can be mapped into its Albanese variety without decreasing its dimension, we can show now that Σ_d has a universal family.

A weaker problem than the existence of a universal family, which seems to be worthwhile to solve nevertheless, is the following. Let X be an ample polar divisor of $\underline{V}$, f_1 a non-degenerate projective embedding of $\underline{V}$ determined by X and $\mathcal{F}_1$ a maximal algebraic family of non-singular projective varieties such that each component $\mathcal{F}_1$ contains $f_1(\underline{V})$. Let $\mathcal{M}_1$ be the quotient space of $\mathcal{F}_1$ which we get by identifying members of $\mathcal{F}_1$ which are isomorphic to each other. Let $X_2 = m_2 X$, where m_2 is a positive integer, and define f_2, $\mathcal{F}_2$, $\mathcal{M}_2$ as f_1, $\mathcal{F}_1$, $\mathcal{M}_1$. When we continue this process, we get a sequence of quotient spaces $\{\mathcal{M}_i\}$ and morphisms $g_i : \mathcal{M}_i \longrightarrow \mathcal{M}_{i+1}$. One can introduce the quotient topology on $\mathcal{M}_i$ and g_i becomes an injection with respect to this topology. Then one can show that there is an open subset $\mathcal{M}'_i$ on each $\mathcal{M}_i$, which is everywhere dense in $\mathcal{M}_i$, such that it has a structure of a union of a finite set of irreducible algebraic varieties. Moreover, g_i induces on each $\mathcal{M}'_i$ a birational morphism such that the closure of the image is $\mathcal{M}_{i+1}$. Then, one could ask if there is a constant c such that g_i is a bijection when $i > c$. When the answer to this is affirmative, we shall call

$\mathcal{F}_m$, for $m > c$, a <u>local universal family at</u> $\underline{V}$.

Once the question of the existence of a universal family of Σ_d, or at least the existence of a local universal family at $\underline{V}$ is settled affirmatively, the study of the space of moduli $\mathcal{M}_{\mathcal{C}}$ (resp. local space of moduli) can be reduced to the study of the quotient space of the universal family (resp. local universal family). But the problem of studying a quotient space of an algebraic variety with respect to an equivalence relation is not a trivial problem. For this purpose, we have to analyze our equivalence relation on a universal family (resp. local universal family) more closely.

THEOREM 2. Let us assume that a universal family of Σ_d (resp. local universal family at $\underline{V}$) exists. Then there exists a universal family (resp. local universal family) $\mathcal{F}$ with the following properties:

(i) $\mathcal{F}$ is a union of a finite set of irreducible maximal algebraic families $\mathcal{F}_i$ in a projective space;

(ii) When $\underline{U}$ is in $\mathcal{F}_i$ and $\underline{W}$ is in $\mathcal{F}$ such that $\underline{U} \sim \underline{W}$, then $\underline{W}$ is in $\mathcal{F}_i$;

(iii) Let Y be a $\underline{U}$-divisor which is algebraically equivalent to a hyperplane section of $\underline{U}$ and call $\mathcal{L}(Y)$ the corresponding invertible sheaf on $\underline{U}$. Then $h^i(\mathcal{L}(Y)) = 0$ for $i > 0$ whenever $\underline{U} \in \mathcal{F}$.

From now on, we shall consider only those universal families (resp. local universal families) which satisfy (i), (ii), (iii) of the above theorem. When that is so, the study of the quotient space of $\mathcal{F}$ can be reduced substantially to those of $\mathcal{F}_i$. In order to do so, we shall assume that Σ_d does not contain a ruled variety. Let F_i be the Chow-variety of $\mathcal{F}_i$ and denote by $0(x)$ the orbit of $x \in F_i$ with respect to our equivalence relation. Then our equivalence relation satisfies the following conditions.

THEOREM 3. (I) The equivalence relation on F_i is a closed equivalence relation; (II) $0(x)$ is an irreducible and locally closed subvariety of F_i; (III) Let k be a field of definition of F_i, x a point of F_i and x' a point of F_i such that $x \longrightarrow x'$ ref. k. Identifying $0(x)$, $0(x')$, with cycles in the ambient projective space, we have $0(x) \longrightarrow m0(x')$ ref. k, where m is a positive integer.

REMARK. Actually, m can be described in terms of the relative change of groups of automorphisms of members of $\mathcal{F}_i$, but we are not going into the detail of this fact.

§2. Nagata has constructed an example of a non-singular locally closed subvariety of a projective space, carrying an equivalence relation which satisfies (I), (II), (III) of Theorem 3, such that the quotient space is not an algebraic variety. On the other hand, we encounter quite often an equivalence relation of this type on an algebraic variety in algebraic geometry. (perhaps omitting the condition that $0(x)$ is irreducible). Moreover, even if the quotient space is an algebraic variety and F_1 is non-singular, it cannot be non-singular in general. Therefore, it seems to be desirable to have some theory which eliminates these difficulties. For this reason, we shall introduce the concept of Q-varieties and Q-manifolds, which can be described briefly as follows.

Let V be an algebraic variety defined over a field k and $\overline{\Gamma}$ a k-closed subset of $V \times V$. When P is a point of V, define $\overline{\Gamma}\{P\}$ by $P \times V \cap \overline{\Gamma} = P \times \overline{\Gamma}\{P\}$. Assume that $(V, \overline{\Gamma})$ has the following properties.

(a) Every component Γ_i of $\overline{\Gamma}$ has the geometric projection V on each factor of the product $V \times V$; $V \leftarrow \Gamma_i \rightarrow V$ dim Γ_i

(b) $\overline{\Gamma}$ defines an equivalence relation on V;

(c) When P and P' are points on V such that P' is a specialization of P over k, $\overline{\Gamma}\{P'\}$ is a uniquely determined specialization of $\overline{\Gamma}\{P\}$ over k over the specialization $P \longrightarrow P'$ ref. k;

(d) When P is a generic point of V over k, every component of $\overline{\Gamma}\{P\}$ is separably algebraic over $k(P)$.

It can be verified easily, using Theorem 3, that the equivalence relation on F_1 satisfies these four conditions.

Let $\mathcal{V}$ be a quotient space of V by this equivalence relation and φ the canonical map of V on $\mathcal{V}$. We make $\mathcal{V}$ a topological space by taking the quotient topology and calling it a Q-<u>variety.</u> Let k be a field of definition of V such that $\Gamma = \Sigma\, \Gamma_i$ is rational over it. k is then called a field of definition of $\mathcal{V}$. When P is a point of V, $\varphi(P)$ is called a point of $\mathcal{V}$. When P' is another point of V such that P' is a specialization of P over k, we say that $\varphi(P')$ is a specialization of $\varphi(P)$ over k. Next, assume that $\bar{\Gamma}\{P\}$ contains a simple point Q on V and set $(Q \times V).\Gamma = Q \times \Gamma(Q)$. $\Gamma(Q)$ is then uniquely determined by $\bar{\Gamma}\{P\}$, i.e., by $\varphi(P)$. Hence we denote it by $\Gamma(\varphi(P))$. Let $\Gamma(\varphi(P)) = \Sigma\, a_i X_i + \Sigma\, b_j Y_j$ be the reduced expression for $\Gamma(\varphi(P))$ such that $a_i \not\equiv 0(p)$ and $b_j \equiv 0(p)$. Denote $\Sigma\, a_i X_i$ by $\Gamma(\varphi(P))_0$ and $\Sigma\, b_j Y_j$ by $\Gamma(\varphi(P))_p$. We call $\varphi(P)$ a <u>regular point</u> of $\mathcal{V}$, and a p-<u>regular point</u> if $\Gamma(\varphi(P))_0 \neq 0$. If $\varphi(P)$ is a p-regular point of $\mathcal{V}$, $\Gamma(P)_0$ has a smallest field K, containing k, over which it is rational. Denote K by $k(\varphi(P))$. It can be shown that this field is also a smallest field, containing k, over which $\Gamma(\varphi(P))$ is rational. If $\varphi(P)$ is not a p-regular point on $\mathcal{V}$, set $\bar{\Gamma}\{P\} = \cup\, Z_i$ and $Z = \Sigma\, Z_i$. Z has a smallest field K', containing k, over which it is rational. We denote K' by $k(\varphi(P))$.

REMARK. When $\varphi(P)$ is p-regular, we could associate K' over k by means of the latter method. It can be shown that K' contains $k(\varphi(P))$ and that the former is a purely inseparable extension of the latter. Moreover, when V is non-singular and $\mathfrak{T}(\varphi(P))_p = 0$, it can be shown that $K' = k(\varphi(P))$.

Using these, the concepts of subvarieties, regular subvarieties, p-regular subvarieties, fields of definitions of these subvarieties and dimensions can be defined as usual. The same is true with the concept of product. Then a point on the product is p-regular if and only if each factor is p-regular. Let $\mathcal{V} \times \mathcal{W}$ be a product of two Q-varieties and $\mathcal{Z}$ a p-regular subvariety of $\mathcal{V} \times \mathcal{W}$ with the projection $\mathcal{Z}'$ on $\mathcal{V}$. The index $[\mathcal{Z} : \mathcal{Z}']$ can be defined in the usual manner. When $\mathcal{Z}' = \mathcal{V}$ and $[\mathcal{Z} : \mathcal{V}] = 1$, we can define a rational map of $\mathcal{V}$ into $\mathcal{W}$. We say that this map is defined at a point ξ if there is a p-regular point $\xi \times \eta$ on $\mathcal{V} \times \mathcal{W}$ such that it is a component of $\xi \times \mathcal{W} \cap \mathcal{Z}$. Using these, we can introduce the concepts of a morphism, a birational correspondence and an isomorphism.

When $\mathcal{V}$ consists entirely of p-regular points, we call it a Q-<u>manifold.</u> When the $\mathcal{V}_i$ are Q-manifolds, finite in number, and the f_{ji} isomorphisms of open subsets of the $\mathcal{V}_i$ into the $\mathcal{V}_j$ such that the graphs of the f_{ji} are closed on the $\mathcal{V}_i \times \mathcal{V}_j$ and that $f_{ji} \circ f_{i\nu} = f_{j\nu}$. Then we can glue the $\mathcal{V}_i$ together by means of the f_{ji} and get an <u>abstract</u> Q-<u>manifold.</u> A subvariety of an abstract Q-manifold may not be an abstract Q-manifold. A subvariety of a Q-manifold may not be a Q-variety. Hence, we define a Q-submanifold of an abstract Q-manifold by means of an abstract Q-manifold and of an injection map. It is on this abstract Q-manifold that we have a complete theory of intersection-multiplicities except for the criterion of multiplicity 1, when we allow the multiplicities to be rational numbers.

Thus, we can deal with a Q-variety as if it is an abstract algebraic variety as far as qualitative problems are concerned. In the same way, we can handle an abstract Q-manifold as if it is a non-singular abstract variety whenever quantitative problems are concerned.

Now it would be clear from Theorems 2 and 3 that the space of moduli (resp. local space of moduli at $\underline{V}$) is a union of a finite set of Q-varieties as soon as a universal damily (resp. local universal family) exists. By Theorem 1 such is the case for polarized surfaces. Moreover, the varieties of moduli of curves and polarized Abelian varieties of bounded rank are Q-manifolds, the latter part of which generalizes Satake's result based on the concept of V-manifolds. Thus, our result could be regarded as a basic step in further development of the problems of moduli. But still at this basic level, there are some interesting unsettled problems which are implicitly contained in this note.

THE BOUNDARY OF MODULI SCHEMES

by David Mumford

1° Discussion

To begin with, what is a variety of moduli ? Start with the set of all non-singular complete varieties of dimension n and arithmetic genus p. For each isomorphism class of these, take one point : then try to put these points together in a variety. There are some more requirements : a "nearby" pair of varieties V_1, V_2 should correspond to a "nearby" pair of points : e.g.

Let $\mathcal{S}$ = set of isomorphism classes of V's

$U \subset \mathcal{S}$ is "open", if for all families of varieties of the given type, varieties of type U occur over an open set in the parameter space.

Another requirement is that for all families

$$\pi : \mathcal{V} \longrightarrow S$$

suppose you map S to $\mathcal{S}$ by assigning to each $s \in S$ the class of the fibre $\pi^{-1}(s)$: then this map should be algebraic.

The problem, in this raw form, has been modified bit by bit so as to make it more plausible :

(I.) Instead of classifying "bare" varieties V, one seeks to classify pairs $(V, \mathcal{D})$ where $\mathcal{D}$ is a numerical equivalence class of very ample divisors on V.

(II) Then break up the set $\mathcal{S}$ via the Hilbert polynomials of the divisors in $\mathfrak{D}$: viz. for every P , let $\mathcal{S}^P$ = isom. classes of $(V, \mathfrak{D})$ such that for all $D \in \mathfrak{D}$

$$P(n) = \chi(\underline{o}_V(nD)) .$$

Now we are close to a good problem :

for all $D \in \mathfrak{D}$

for all bases of $H^0(V, \underline{o}_V(D))$ you get a canonical immersion

$$V \subset \mathbb{P}_n \qquad (n = \dim H^0(V, \underline{o}_V(D)) - 1)$$

s.t. hyperplane sections are linearly equivalent to D .

i.e. $$\mathcal{S}^P \cong \frac{\text{certain set of subvarieties } V \text{ of } \mathbb{P}_n}{\text{certain equivalence relation, especially projective equivalence}}$$

(III) . Why insist that V be non-singular ? The only reason appears to be that over $\mathbb{C}$ families of non-singular varieties are locally differentiably trivial : so one can view them as families of complex structures on a fixed differentiable manifold, (or, as in the Bers-Ahlfors approach, on a fixed topological manifold) . Algebraically, there is no point : let's let V be any complete variety at all, maybe even reducible and assume that $\mathfrak{D}$ is a class of Cartier divisors.

To go further, let's stop and ask what problems arise : first we should take a broad look at the topology which we are getting by throwing in all varieties - typically it will be very un-separated ; second we should try to find open subsets $U \subset \mathcal{S}^P$ such that, in their induced topology, they are separated, and "compact" if possible.

[This means that U U could be given the structure of a moduli variety, it would turn out complete ; and it also means, directly, that if $(V, \mathfrak{D}) \in U$, and we specialize the groundfield, then we can find a specialization $(\overline{V}, \overline{\mathfrak{D}})$ of $(V, \mathfrak{D})$ also in U.]

Thirdly, we will finally have to find out if U can be made into a <u>variety.</u>

(IV.) We understand the last problem better when we realize that, e.g. via chow coordinates, <u>almost all</u> of U is bound to come out as a variety. We saw that $\mathcal{S}^P$ was a quotient of a piece $\mathcal{H}$ of the chow variety by an algebraic equivalence relation. Such quotients always exist <u>birationally,</u> i.e. for a small enough Zariski-open subsets $U^* \subset \mathcal{H}$, [U^*/modulo equivalence relation] will be a good variety. So the 3^{rd} problem is like the first two :

<u>The only problem is to pick the "boundary" components shrewdly,</u> i.e. to decide which non-generic varieties to allow .

there again, it would prejudice the issue to think that we should necessarily use all and/or only non-singular varieties. And the choice should be made by a) checking the topology and b) checking its "algebraizability".

(V.) A final step in setting up the problem reasonably is to realize that all the same questions occur equally well for a much more general class of problems : viz. that of forming quotients of varieties by algebraic equivalence relations. Only by realizing this can we hope to find simple enough examples to study first so as to get the right feeling. Especially, the hard equivalence relations are the non-compact one's ; and in the case of moduli, this occurs principally in forming :

$$\mathcal{H} / \{ \text{Projective equivalence of } V\text{'s in } \mathbb{P}_n \}$$

i.e. in forming an orbit space by PGL (n) .

2° Present State of the Theory

very good (i) analogous problem in classifying vector bundles on a fixed curve

pretty good (ii) moduli of curves (canonically polarized)

half good (iii) moduli of polarized abelian varieties

no good (iv) moduli of surfaces of general type

3° An Example

Rather than analyze an actual moduli problem, I want to take one of the simplest non-trivial orbit space problems, in which all the features of the conjectured results occur :

$G = PGL(1)$ acting on $\mathbb{P}_n$, where $\mathbb{P}_n = n^{th}$ symmetric product of $\mathbb{P}_1$, i.e. $PGL(1)$ acting on the set of 0-cycles of degree n.

(= theory of binary quantics) .

a) <u>jump phenomenon</u>

look at $\mathbb{P}_2/PGL(1)$. There are 2 orbits : $\{P+Q \mid P \neq Q\}$ and $\{2P\}$. Therefore, get 2pts. x, y where x is open but not closed, y is closed but not open :

$$\bullet \longrightarrow \bullet$$

This occurs in all moduli problems, and one always must exclude some points to avoid this.

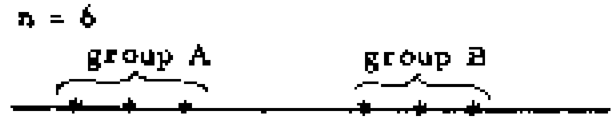
In $\mathbb{P}_n$, exclude the 0-cycles

$$kP + (n-k)Q$$

whose isotropy group is infinite.

b) <u>further non-separation</u>

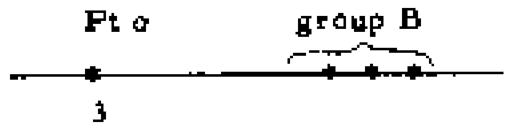
take $n = 6$



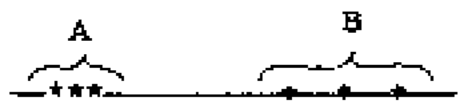
generic cycle.

Let all points in group A come together ; you get in the limit :

(*)



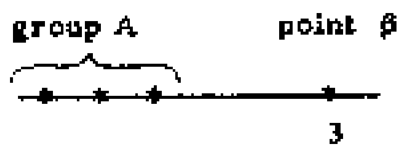
But suppose, as group A collapses to α, you apply a one-parameter subgroup $G_m \subset PGL(1)$, moving points away from α to β. Then the following are **projectively-equivalent:**



and

the latter approaches :

(**)



But the 0-cycles (*) and (**) are probably not projectively equivalent.

c) <u>the unitary retraction</u> : to avoid these bad things, define

$$\mathcal{K} \subset \mathbb{P}_n$$

$\mathcal{K}$ = Set of 0-cycles $\sum_{i=1}^{n} P_i$, such that, putting the P_i on the Gauss sphere, and embedding the Gauss sphere in $\mathbb{R}^3$ as $x^2 + y^2 + z^2 = 1$, then the <u>vector</u> sum of the P_i in $\mathbb{R}^3$ is $(0, 0, 0)$.

One checks, if $x, y \in \mathcal{K}$, then x, y are equivalent under PGL(1) if and only if they are equivalent under the maximal compact subgroup

$$K = SO(3; \mathbb{R}) \subset PGL(1, \mathbb{C}) = G .$$

But $\mathcal{K}$ is compact, therefore $\mathcal{K}/K$ is compact and separated. And

$$\mathcal{K} \cdot PGL(1) = \left\{ \mathfrak{A} \;\middle|\; \begin{array}{l} \text{no point } Q \text{ occurs in with} \\ \text{multiplicity} > n/2 \text{; and if } Q \text{ occurs} \\ \text{with multiplicity } n/2 \text{, then} \\ \mathfrak{A} = \frac{n}{2}(Q+Q') \end{array} \right\} .$$

d) <u>stability restriction</u> : $\mathcal{K} \cdot PGL(1)$ contains a Zariski-open set

$$U_{stable} = \left\{ \mathfrak{A} \;\middle|\; \begin{array}{l} \text{no point } Q \text{ occurs in } \mathfrak{A} \text{ with} \\ \text{multiplicity} \geqq n/2 \end{array} \right\}$$

So U_{stable}/G has separated topology, and is compact if n is odd. It is also a variety by virtue of a general theorem of mine.

e) <u>semi-stability</u> : when n is even, things are less clean. $\mathcal{K}$ showed that there was a natural compactification of U_{stable}/G by adding a <u>single</u> point representing the cycles $n/2(Q+Q')$. In fact, there is a complete <u>variety</u> $\overline{V}_n$, with point ∞ and diagram of algebraic maps:

$$\begin{array}{ccc} U_{\text{semi-stable}} & \longrightarrow & \bar{V}_n \\ \bigcup & & \bigcup \\ U_{\text{stable}} & \longrightarrow\ U_{\text{stable}}/G \ = & \bar{V}_n - (\infty) \end{array}$$

where

$$U_{\text{semi-stable}} = \left\{ \mathfrak{A} \,\middle|\, \begin{array}{l}\text{no point } Q \text{ occurs in } \mathfrak{A} \text{ with} \\ \text{multiplicity } > n/2\end{array} \right\} .$$

D. N. Clan

Invariants of a group in an affine ring

by

Masayoshi NAGATA

1. When a group G acts on a ring R inducing a group of automorphisms, then we can speak of G-invariants in R. Let us denote the set of G-invariants in R by $I_G(R)$. Our particular interest lies in the case where R is a finitely generated commutative ring over a field K and the action of G on R is such that 1) the automorphisms are K-isomorphisms and 2) $\Sigma_{g \in G} f^g K$ is a finite K-module for every $f \in R$. In this case, let $f'_1, \ldots, f'_{n'}$ be a set of generators of R over K and choose a linearly independent base $f_1, \ldots, f_n$ of $\Sigma_i (\Sigma_{g \in G} (f'_i)^g K)$. Then $R = K[f_1, \ldots, f_n]$ and the action of F on R is characterized by the representation of G defined by the module $\Sigma_{i,g} f_i^g K$. Thus, in order to observe $I_G(R)$, we may assume that

(1) G is a matric group contained in $GL(n, K)$, and

(2) $R = K[f_1, \ldots, f_n]$ and, for every $g \in G$, the automorphism of R defined by g is induced by the linear transformation

$$\begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} \longrightarrow g \begin{pmatrix} f_1 \\ \cdot \\ \cdot \\ \cdot \\ f_n \end{pmatrix} .$$

Under the circumstance, the following results are known :

LEMMA 1.1. $I_G(R)$ is finitely generated if every rational representation of G is completely reducible or if G is a finite group, hence if G has a normal subgroup N of finite index such that every rational representation of N is completely reducible.

In the general case, there are some examples of a pair of G and R such that $I_G(R)$ is not finitely generated.

LEMMA 1.2. If $\bar{G}$ is the smallest algebraic set in $GL(n, K)$ among those containing G, then $\bar{G}$ is a group which acts on R naturally and $I_G(R) = I_{\bar{G}}(R)$.

LEMMA 1.3. If K' is a ring containing K, then, under a natural extension of the action of G on $R \otimes_K K'$ such that every element of K' is G-invariant, we have $I_G(R \otimes_K K') = I(R) \otimes_K K'$.

By virtue of Lemmas 1.2, 1.3 above, we see that, in asking finite generation of $I_G(R)$, fundamental is the case where G is an algebraic group with universal domain K. But, such an assumption does not bring us any simplicity in our treatment. Therefore we shall not assume that G is an algebraic group, but assume the assumptions (1) and (2) above.

Furthermore, rational representations of G which we meet in our treatment are rather special, and therefore it is good enough to understand by a rational representation of G a representation obtained in the following manner :

Let R^* be the polynomial ring over K in indeterminates $X_1, \ldots, X_n$. Then G acts on R^* as defined by

$$\begin{pmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{pmatrix} \longrightarrow g \begin{pmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_n \end{pmatrix} \quad \text{for each } g \in G .$$

Let M and N be G-stable finite K-modules contained in R^* such that $N \subseteq M$. M/N defines a rational representation of G. Rational representations we shall meet with in this paper are those of this type.

2. We call G a <u>reductive group</u> if every rational representation of G is completely reducible. It is known that

LEMMA 2.1. If G is an algebraic group, then (i) in the characteristic zero case, the reductivity is equivalent to the condition that the radical is a torus and (ii) in the case of characteristic $p \neq 0$, the reductivity is equivalent to the condition that the connected component G_0 of the identity of G is a torus and furthermore the index $(G : G_0)$ is prime to p.

Thus the class of reductive groups is not very small in the characteristic zero case, but is very small in the positive characteristic case. Thus, in view of the known counter-example to the 14-th problem of Hilbert, the following consequence of Lemma 1.1 is rather satisfactory in the characteristic zero case and is very unsatisfactory in the positive characteristic case :

LEMMA 2.2. In the characteristic zero case, $I_G(R)$ is finitely generated if the radical of the smallest algebraic group $\bar{G}$ in $GL(n, K)$ among those containing G is a torus : in the positive characteristic case, $I_G(R)$ is finitely generated if the connected component of the identity of $\bar{G}$ is a torus.

3. Let us denote by P_m from now on the polynomial ring over K in n indeterminates $X_1, \ldots, X_m$.

Let ρ be a rational representation of G. If $\rho(G) \subseteq GL(m, K)$, then we define an action of G on P_m by

$$\begin{pmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{pmatrix} \to \rho(g) \begin{pmatrix} X_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_m \end{pmatrix} \quad \text{for every } g \in G .$$

We call G a **semi-reductive group** if the following is true : If ρ is a rational representation of G which defines an action on P_m (m being such that $\rho(G) \subseteq GL(m, K)$) such that (i) $\Sigma_{i \geq 2} X_i K$ is G-stable and (ii) X_1 modulo $\Sigma_{i \geq 2} X_i K$ is G-invariant, then there is a polynomial $F \in P_m$ which is G-invariant, monic in X_1 and of positive degree in X_1.

Since the action of G preserves the degree of every homogeneous form, the condition on F above may be replaced by the condition to be a G-invariant homogeneous form of positive degree which is monic in X_1.

For algebraic linear groups, it was conjectured by D. Mumford that if the radical is a torus then the group is semi-reductive. As will be shown below, this conjecture is equivalent of the following, which we like to call Mumford Conjecture :

Mumford Conjecture. If G is a connected semi-simple algebraic linear group, then G is semi-reductive.

To the writer's knowledge, Mumford Conjecture has been solved only in a very special case where characteristic is 2 and $G = SL(2, K)$; it was done by Mr. Tadao Oda.

The purpose of the present note is to show

MAIN THEOREM. $I_G(R)$ is finitely generated if G is semi-reductive.

Let us indicate here how to prove the equivalence of Mumford conjecture with the case of an algebraic group whose radical is a torus. The key lemma is :

LEMMA 3.1. Let N be a normal subgroup of G. If both N and G/N are semi-reductive, then G is also semi-reductive.

Proof: Let ρ be a rational representation of G as stated in the definition of semi-reductivity. Then the restriction ρ' of ρ on N is of the same type, whence there is a homogeneous form $F \in P_m$ of positive degree such that F is monic in X_1 and N-invariant under the action of N defined by ρ'. Consider the G-module $M = \Sigma_{g \in G} F^g K$. The action of G on M is really an action of G/N. Let M^* be $M \cap \Sigma_{i \geq 2} X_i P_m$, and let $F_1, \ldots, F_s$ be a base of M^*. Then, since $M = FK + M^*$, since any power of X_1 is G-invariant modulo $\Sigma_{i \geq 2} X_i P_m$, the semi-reductivity of G/N implies the existence of a homogeneous form F^* in $F, F_1, \ldots, F_s$ of positive degree such that (i) it is monic in F and (ii) it is G-invariant. F^* is a homogeneous form of positive degree in $X_1, \ldots, X_m$. Since $F_i \in \Sigma_{j \geq 2} X_j P_m$ and since F is monic in X_1, we see that F^* is monic in X_1. Thus G is semi-reductive.

Now the equivalence said above is proved easily by the fact that finite groups and tori are all semi-reductive.

4. Before proving our main theorem, we like to give a remark on our formulation of Mumford Conjecture. Mumford's formulation was stated in projective space. Namely, if ρ is a rational representation of G and if $\rho(G) \subseteq GL(m, K)$, then an action of G on P_m is defined, which defines an action of G on the projective space S^{m-1} of dimension $m-1$. The condition proposed by Mumford is that if a point $P \in S^{m-1}$ is G-invariant, then there is a G-stable hypersurface in S^{m-1} which does not go through P.

If this condition is stated in P_m, then, choosing coordinates of P to be $(1, 0, \ldots, 0)$, it can be stated as follows:

If $\Sigma_{i \geq 2} X_i K$ is G-stable (hence, X_1 modulo $\Sigma_{i \geq 2} X_i K$ is G-semi-invariant), then there is a G-semi-invariant homogeneous form F which is monic in X_1 and of positive degree.

PROPOSITION 4.1. If the above condition is satisfied by G, then G is semi-reductive.

Proof: Let ρ be as in the definition of semi-reductivity. Then there is a homogeneous form F as in the above condition. Since X_1 is invariant modulo $\Sigma_{i \geq 2} X_i K$ under the action of G, any power of

X_1 is G-invariant modulo the ideal generated by $\Sigma_{i \geq 2} X_i K$. Therefore that F is G-semi-invariant implies that F is G-invariant.

The converse of Proposition 4.1 is also true, and was proved by Mr. M. Miyanishi. The proof will be given at the end of this article as an appendix.

5. A reductive group is obviously a semi-reductive group, hence our main theorem includes the corresponding result for reductive groups. As for the proof, that special case is much simpler than the semi-reductive case. In order to compare these cases, let us begin with glance at the reductive case.

The following two are key lemmas to prove our main theorem for reductive groups :

LEMMA 5.1. A. Let ϕ be a G-homomorphism from R onto a ring R'. If G is reductive, then $I_G(R') = \phi(I_G(R))$.

LEMMA 5.2. A. If G is reductive, then for any $h_1, \ldots, h_s$ in $I_G(R)$, we have $(\Sigma_i h_i R) \cap I_G(R) = \Sigma_i h_i(I_G(R))$.

Namely, the first lemma enables us to assume that $f_1, \ldots, f_n$ are algebraically independent. Then the second lemma shows that $I_G(R)$ is a graded Noetherian ring, and we see easily that $I_G(R)$ is finitely generated, by virtue of a well known lemma which will be recalled later.

For semi-reductive groups, we have the following adaptions of the above lemmas :

LEMMA 5.1. B. With the same notations as above, if G is semi-reductive, then, for every element x of $I_G(R')$, there is a power x^t of x such that $x^t \in \phi(I_G(R))$. Consequently, $I_G(R')$ is integral over $\phi(I_G(R))$ in this case.

LEMMA 5.2. B. Assume that G is semi-reductive. Then for any $h_1, \ldots, h_s \in I_G(R)$, every element of $(\Sigma_i h_i R) \cap I_G(R)$ is nilpotent modulo $\Sigma_i h_i (I_G(R))$.

<u>Proof of Lemma 5.1.B</u>: Let y be an element of R such that $\phi(y) = x$. Set $M = \Sigma_{g \in G} y^g K$, $\mathfrak{a} = \phi^{-1}(0)$, $N = M \cap \mathfrak{a}$. If $x = 0$, then the assertion is obvious, and we assume that $x \neq 0$. Since x is G-invariant, we have $y^g - y \in N$ for every $g \in G$. Therefore, letting $y_1, \ldots, y_m$ be a linearly independent base of N, we see that, by virtue of the semi-reductivity of G, there is a G-invariant element F of $K[y, y_1, \ldots, y_m]$ which is monic and of positive degree, say t, in y, and homogeneous in $y, y_1, \ldots, y_m$. Then $\phi(F) = x^t \in \phi(I_G(R))$. This completes the proof of Lemma 5.1.B.

Proof of Lemma 5.2.B. We shall make use of induction argument on s without fixing R. Let ϕ be the natural homomorphism from R onto R/h_1R. Let x be an arbitrary element of $(\Sigma_i h_iR) \cap I_G(R)$. Then $\phi(x)$ is in $\Sigma_{i \geq 2}\phi(h_i)\phi(R) \cap \phi(I_G(R))$, whence by induction on s, we see that there is a natural number t such that $\phi(x^t)$ is in $\Sigma_{i \geq 2}\phi(h_i)I_G(\phi(R))$. This means that $x^t = \Sigma_i h_i F_i$ with $F_1 \in R$ and $F_2, \ldots, F_s \in \phi^{-1}(I_G(\phi(R))$. By Lemma 5.1.B, there is a natural number u such that $\phi(F_s u) \in \phi(I_G(R))$. Then, considering x^{tu} instead of x^t, we may assume that $F_s \in I_G(R)$ (if $s > 1$). Then $X^t - h_s F_s \in (\Sigma_{i \leq s-1} h_iR) \cap I_G(R)$, and $x^t - h_s F_s$ is nilpotent modulo $\Sigma_{i \leq s-1} h_i(I_G(R))$, which implies the assertion. Therefore we have only to prove the case where $s = 1$. In this case, $x = h_1 x'$ with $x' \in R$ and x' is G-invariant modulo $0 : h_1 R$. Let σ be the natural homomorphism $R \longrightarrow R/(0 : h_1 R)$. Then $\sigma(x') \in I_G(\sigma(R))$, whence there is a natural number t such that $\sigma(x'^t) \in \sigma(I_G(R))$. Let $z \in I_G(R)$ be such that $\sigma(z) = \sigma(x'^t)$. Then $x^t = h_1{}^t x'^t = h_1{}^t z \in h_1(I_G(R))$. This completes the proof of Lemma 5.2.B.

We recall here the lemma on graded Noetherian ring refered above :

LEMMA 5.3. Assume that a ring A is such that (i) it is the direct sum of submodules $A_0, A_1, \ldots, A_n, \ldots$ and (ii) $A_iA_j \subseteq A_{i+j}$ for every possible pair (i, j). If the ideal $\Sigma_{i \geq 1} A_i$ has a finite basis, then A is finitely generated over A_0.

6. Let ϕ be the homomorphism from P_n onto R such that $\phi(X_i) = f_i$ for every i and let k be the kernel of ϕ. We shall prove here the main theorem in the case where k is a homogeneous ideal. Since P_n is Noetherian, we can use induction argument on the largeness of k. Thus we assume that if k' is a G-stable homogeneous ideal of P_n and contains k properly, then $I_G(P_n/k')$ is finitely generated.

LEMMA 6.1. Under the circumstance, if h is a graded ideal $\neq 0$ of R, then $I_G(R)/(h \cap I_G(R))$ is finitely generated.

<u>Proof:</u> By assumption, $I_G(R/h)$ is integral over $I_G(R)/(h \cap I_G(R))$. These two facts show the result.

Therefore, by virtue of Lemma 5.3, if there is such an ideal h (not containing 1) as above so that $h \cap I_G(R)$ has a finite basis, then we see the finite generation of $I_G(R)$.

As a particular case, we have the case of an integral domain. Namely if h is a homogeneous element of $I_G(R)$ and if R is an integral domain, then $hR \cap I_G(R) = h(I_G(R))$. The same reasoning is applied if there is a homogeneous element h of positive degree which is not a zero divisor.

Next we consider the case where R is not an integral domain. Let $h \neq 0$ be a homogeneous element of $I_G(R)$ of positive degree. Set $\mathfrak{a} = 0 : hR$. If $\mathfrak{a} = 0$, then we finished already, and we assume that $\mathfrak{a} \neq 0$. Then, by Lemma 6.1, both $I_G(R)/(hR \cap I_G(R))$ and $I_G(R)/(\mathfrak{a} \cap I_G(R))$ are finitely generated. Therefore there is a finitely generated subring A of $I_G(R)$ such that $I_G(R)/(hR \cap I_G(R)) = A/(hR \cap A)$ and such that $I_G(R)/(\mathfrak{a} \cap I_G(R)) = A/(\mathfrak{a} \cap A)$. Since $I_G(R/\mathfrak{a})$ is a finite module over $A/(\mathfrak{a} \cap A)$, there are elements $c_1, \ldots, c_t$ of R such that $I_G(R/\mathfrak{a})$ is generated by these c_i modulo $\mathfrak{a}$ as an $A/(\mathfrak{a} \cap A)$-module. We like to show that $I_G(R)$ is then generated by $c_i h$ over A. Since c_i modulo $\mathfrak{a}$ are G-invariant, we see that $c_i h$ are G-invariant. Conversely, let x be any element of $I_G(R)$. Then there is an element a of A such that $x - a \in hR$. Let r be such that $x - a = hr$ $(r \in R)$. Since hr is G-invariant, we see that r modulo $\mathfrak{a}$ is G-invariant, whence there is an element b of $\Sigma A c_i$ such that $r - b \in \mathfrak{a}$. Then $hr = hb \in A\{hc_1, \ldots, hc_t\}$ this completes the proof, provided that the kernel $\mathfrak{k}$ of ϕ is homogeneous.

7. Now we consider the general case. We adapt the notation is § 6 without assuming that $\mathfrak{k}$ is homogeneous. The induction argument is also adapted, considering all G-stable ideals of P_n. Then we need a different proof only in the case where $I_G(R)$ is an integral domain (for, otherwise, take an element h of $I_G(R)$ which is a zero-divisor in

$I_G(R)$, and adapt the proof just above). In this case, $I_G(R)$ is integral over $I_G(P_n)/((k) \cap I_G(P_n))$. Since the result in § 6 includes the case where $k = 0$, we see that $I_G(P_n)$ is finitely generated, hence the integral dependence implies that $I_G(R)$ is finitely generated. Thus the proof of the main theorem is completed.

APPENDIX

by Masayoshi Miyanishi

We shall prove here the converse of Proposition 4.1 above. Assume that a rational representation ρ of G is of the form

$$\begin{pmatrix} t & \sigma \\ 0 & \rho' \end{pmatrix}$$

where t is of degree 1. Let m be the degree of ρ. Then we consider a representation $\tau = tE$, E being the unit matrix of degree m. Then $\tau(g)$ is in the center of $GL(m, K)$ for every $g \in G$, and therefore $\rho\tau^{-1}$ gives a rational representation of G (not in the restricted sense above, but in the usual sense). By the semi-reductivity of G, there is a homogeneous form F in P_m of positive degree such that it is monic in x_1 and G-invariant under the action of G defined by $\rho\tau^{-1}$. Then F is semi-invariant under the action of G defined by ρ. This proves the converse of Proposition 4.1.

TRANSFORMATION SPACES, QUOTIENT SPACES, AND SOME CLASSIFICATION PROBLEMS.

Maxwell Rosenlicht

For simplicity let us restrict our attention to varieties in the classical sense. If V is a variety and $R \subset V \times V$ is an equivalence relation among the points of V, by a <u>quotient variety</u> is meant a pair $(V/R, p)$, where V/R is a variety and $p : V \longrightarrow V/R$ is a surjective morphism such that two points of V have the same image under p if and only if they are R-equivalent and such that, for any $v \in V$, if f is a rational function on V that is defined at v and is R-invariant (i.e., constant on R-equivalence classes) then f is the composition of a rational function on V/R that is defined at px and the map p [1, exposé 8]. If V/R exists, it clearly satisfies a universal mapping property for R-invariant morphisms of V and, in particular, is essentially unique. However V/R need not exist: one necessary condition for the existence of V/R is that R be a closed subset of $V \times V$.

In what follows, we consider only the case where V is a transformation space for an algebraic group G and

$$R = \{ (v, gv) \mid v \in V, g \in G \}$$

is the equivalence relation whose equivalence classes are the G-orbits on V: in this case it is customary to write V/G instead of V/R

(if this exists). If V/G exists, the map $p : V \longrightarrow V/G$ is automatically separable, for the function field on V/G is the subfield of the function field on V consisting of all elements left fixed by a group of automorphisms. In general, the graph of the operation of G on V is a closed subset of $G \times V \times V$ so that R, the projection of this graph on $V \times V$, is always constructible. The isotropy groups (stability groups) of the points of V can be obtained by intersections on $G \times V \times V$, hence have the obvious semicontinuity property that the dimension of the isotropy subgroup of $v \in V$ is constant for v on a certain G-invariant open subset of V, and greater than this constant on the complementary closed subset. Any given point of V has an orbit and an isotropy group the sum of whose dimensions is dim G, so that all orbits on a certain G-invariant dense open subset of V have the same dimension, and all other orbits have strictly smaller dimension. If it should happen that all orbits have the same dimension, then the fact that the closure of an orbit is also G-invariant would imply that all orbits are closed. However all orbits may be closed without equidimensionality holding; e.g., if G is unipotent and V affine, orbits need not have the same dimension but they are always closed [3]. If R is closed then the equation

$$v \times Gv = R \cap (v \times V)$$

implies that all orbits on a dense open G-invariant subset of V have constant dimension, with other orbits having larger dimension; thus if R is closed, in particular if V/G exists, all orbits are closed and have equal dimension.

If a quotient $p : V \longrightarrow V/G$ exists, a number of other pleasant consequences follow without any further assumption [3]. In this case the map p is open, so that V/G has the expected quotient topology. If $V' \subset V$ is open and G-invariant then the subset V'/G of V/G is a quotient variety of V'. If W is any variety and G operates on $V \times W$ by the rule $g(v, w) = (gv, w)$, then $(V \times W)/G$ exists and equals $(V/G) \times W$. There is not much of a theory on fields of definition, for if G and V are defined over the field k and if V/G is quasi-projective (a condition that can be relaxed somewhat), then V/G and p may both be chosen so as to be defined over k.

The existence of a quotient V/G turns out to be largely a local problem, for if V is covered by G-invariant open subsets $\{V_i\}$ such that each V_i/G exists, then V/G exists if and only if R is closed. But the closure of R does not insure the existence of V/G, as an example of Nagata shows [2]. Good local criteria for the existence of quotients are much to be desired. The most general result in this direction is due to Seshadri [8]. The most important case of Seshadri's result is when V is a principal transformation variety for G, i.e., when R is closed, all isotropy groups are points, and the map $R \longrightarrow G$ given by $(v, gv) \mapsto g$ is a morphism, and it says that if V is normal then each of its points has a G-invariant open neighborhood which has a finite galois covering which is also a principal space for G and in addition admits a quotient by G (so that the existence of V/G depends locally on the existence of quotients for finite groups operating on other varieties). Seshadri has also shown [9] that if V is a normal principal space for an abelian variety G then V/G always exists.

As might be expected, the simplest general result on the existence of quotient varieties is also one of the most useful. It is to the effect that for any transformation space V for the algebraic group G there exists a dense G-invariant open subset V' of V such that V'/G exists [6]. The proof consists in first constructing V/G and p birationally, by means of the G-invariant rational functions on V, and then cutting off closed subsets that cause trouble. In case R is closed, it is immediate that there exists a unique maximal G-invariant open subset V' of V such that V'/G exists. If there exist sufficiently many G-automorphisms of V then V/G will exist, a result which produces a very easy proof of the existence of coset spaces for subgroups of groups, together with all the desired structure and rationality properties of these quotients [4].

There are a number of important results stating that if V is affine and certain other conditions hold then V/G exists and is also affine. In such cases the coordinate ring on V/G must consist of all G-invariant functions in the coordinate ring of V, which gives the starting point of all the proofs, and practically the whole proof in the case where G is finite [7, pp. 57 - 59]. (If G is finite there is an immediate generalization giving the existence of V/G where V is not affine, but each orbit on it is contained in an affine open subset [7]; an example of Nagata for $G = Z/2Z$ shows that this result may fail without the last condition.) The result holds whenever G is a torus and orbits are equi-dimensional [3], and also if G is reductive and all orbits are closed, at least in the case of characteristic zero (Borel, Iwahori, Mumford, Nagata).

There are interesting problems connected with the classification of transformation spaces for algebraic groups, even in the special case where the transformation space is homogeneous or prehomogeneous (i.e., has a dense homogeneous subset) and the group is connected and solvable. If V is homogeneous and G is commutative then, fixing a point of V, V is simply an algebraic group that is a homomorphic image of G, while if G is connected, solvable and linear, then V is isomorphic (as an algebraic set) to a product of affine lines and affine lines with single points deleted [5]. In the last case, if G and V are defined over a field k such that G is k-solvable (meaning, roughly, that G has a composition series over k with all quotient groups isomorphic to the additive or multiplicative group in one variable), then this product decomposition of V can be done rationally over k. In the special case where $\dim G = 1$ the result, even without the rationality part, leads to an easy proof that for any quotient variety $V \longrightarrow V/G$, where G is connected, solvable, and linear, there exists a rational cross-section

$V/G \longrightarrow V$.

The problem of classifying all complete prehomogeneous spaces for connected unipotent groups derives its main interest from the fact that if B is a Borel subgroup of a connected linear algebraic group G then G/B is prehomogeneous for B_u (and furthermore there are only a finite number of orbits, each isomorphic to an affine space). In the same way the operation of a maximal torus T of G on G/B leads one to consider in full generality projective varieties V that are transformation spaces for a torus T, theorems which enable one to read off a good deal of the classification theory of linear algebraic groups [1, exposé 10 ff.]. For example, one can prove easily that the fixed points for T on V are at least $\dim V + 1$ in number and all of V is left fixed by a subtorus of T of codimension $\leq \dim V$, which two facts together almost imply that a semisimple linear algebraic group is generated by its 3-dimensional simple subgroups.

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ON THE THEORY OF COMPACTIFICATIONS

Jun-Ichi Igusa

This is the first part of our lecture, "On the Siegel modular variety", and it contains an outline of a proof of the fact that compactification of Satake's type [1] have, under certain general conditions, no finite non-singular coverings locally at the boundary points. This fact was observed in the case of the compactifications of the Siegel upper-half plane of genus two [5, cf. 7]. However, the proof we had in that case was too special. Following a suggestion given to us by Zariski, with the use of our results on "theta-constants" we then examined the compactification of the Siegel upper-half plane of arbitrary genus and found the under-lying mechanism, which we find convenient to explain using the theory of "Siegel domains of the third kind" developed by Pyatetski-Shapiro [10]. We shall, therefore, start summarizing Pyatetski-Shapiro's results (making a minor correction) to increase the readability.

1. Let T be a bounded domain, i.e., a non-empty bounded connected open subset of a complex vector space, or at least (complex) analytically isomorphic to a bounded domain, and let U, Z be complex

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1. This means the compactifications of quotient varieties of bounded symmetric domains (by certain properly discontinuous groups of analytic automorphisms) which are obtained by "adding" quotient varieties of some boundary components (using Cartan's theorem on the prolongation of normal analytic spaces [2]). A most general theory of compactifications (of Satake's type) has recently been obtained by Baily and Borel.

vector spaces all three of finite dimensions. Let R be a "real subspace" of Z, i.e., a subspace of Z when Z is considered as a vector space over $\mathbb{R}$, such that Z splits into a direct sum of R and iR : $Z = R + iR$, $R \cap iR = 0$. If z is a vector in Z, it can be written uniquely in the form $\mathrm{Re}(z) + i\mathrm{Im}(z)$ with $\mathrm{Re}(z)$, $\mathrm{Im}(z)$ in R. We shall assume that a non-empty open convex cone C which contains no entire straight line is given in R. This means that, with respect to a suitable affine coordinate system in R, C is contained in the first quadrant. We shall assume that, for every point t of T, a "quasi-hermitian form" $L_t : U \times U \longrightarrow Z$ is given. This means that $L_t(u,v)$ is $\mathbb{C}$-linear in u, $\mathbb{R}$-linear in v and

$$[u,v] = (1/2i)(L_t(u,v) - L_t(v,u))$$

is "real", i.e., R-valued. We then consider the set S of points in the product $Z \times U \times T$ with coordinates (z,u,t) satisfying

$$\mathrm{Im}(z) - \mathrm{Re}(L_t(u,u)) \in C .$$

We shall impose further conditions. We require first that the mapping $U \times T \longrightarrow R$ given by $(u,t) \longrightarrow \mathrm{Re}(L_t(u,u))$ is continuous. This implies that S is an open subset of $Z \times U \times T$. We then require that S is analytically isomorphic to a bounded domain. The third condition is more involved. We consider the set $\mathcal{U}$ of analytic mappings $b : T \longrightarrow U$ such that the mapping $T \longrightarrow Z$ defined by $t \longrightarrow L_t(u, b(t))$ is also analytic for every u in U. This implies that the mapping $T \times T \longrightarrow Z$ defined by $(t,t') \longrightarrow L_t(b(t'), b(t))$ is analytic. At any rate, it is clear that $\mathcal{U}$ forms a vector space over $\mathbb{R}$. We require that $\mathcal{U}$ and U have the same dimension over $\mathbb{R}$. As it was shown by Pyatetski-Shapiro, this means that, if t_0 is an arbitrary point of T , the mapping $\mathcal{U} \longrightarrow U$ defined by $b \longrightarrow b(t_0)$ is an isomorphism over $\mathbb{R}$. If these conditions are satisfied, we say that S is a Siegel domain (of the third kind) over T .

Suppose that S is a Siegel domain over T . Then, for (b, a) in $\mathcal{U} \times R$, the mapping $(b, a) : S \longrightarrow S$ defined by

$$(z, u, t) \longrightarrow (z + a + iL_t(2u + b(t), b(t)) \, , \, u + b(t) \, , \, t)$$

is an analytic automorphism of S . These automorphisms of S form a subgroup G_3 of the group G_0 of all analytic automorphisms of S . The law of composition in G_3 is given by

$$(b, a)(b', a') = (b + b', \; a + a' + 2[b, b']) \, .$$

We note , here, that the mapping $[b, b'] : T \longrightarrow Z$ can be identified with an element of R , because it is analytic and R-valued, hence constant. We also make the following observation. Consider the fiber over t , say S_t , of the fibering $S \longrightarrow T$ defined by $(z, u, t) \longrightarrow t$. Consider further the fibering $S_t \longrightarrow C$ defined by $(z, u, t) \longrightarrow \mathrm{Im}(z) - \mathrm{Re}(L_t(u, u))$. This fibering has a global cross-section defined by $r \longrightarrow (ir, 0, t)$ and the fiber bundle $S_t \longrightarrow C$ is isomorphic to the product-bundle $G_3 \times C \longrightarrow C$ in an obvious way. Since G_3 operates on each fiber as left translations, it is called the group of translations in S . In the following, we shall assume that the skew-symmetric bilinear form $[b, b']$ is non-degenerate. It is the same thing to assume that the center G_4 of G_3 is the subgroup defined by $b = 0$.

This assumption is always satisfied in the applications. Since G_4 is isomorphi to $\mathbb{R}$ as $a \longrightarrow (0, a)$, it will be identified with $\mathbb{R}$. Then we have an isomorphism $G_3/\mathbb{R} \simeq \mathcal{U}$ induced from $(b, a) \longrightarrow b$. We recall that, at each point t_0 of T, we have an isomorphism $\mathcal{U} \simeq U$ over $\mathbb{R}$ defined by $b \longrightarrow b(t_0)$. If Γ is a subgroup of G_3, we shall denote its image in U under the composite mapping simply by $\Gamma(t_0)$.

We shall also define G_1 and G_2. Consider the group of analytic automorphisms of S of the following form

$$(z, u, t) \longrightarrow (Az + a(u, t), B(t)u + b(t), g(t)) ,$$

in which g represents analytic automorphisms of T. They form a subgroup of G_0, and this is G_1. As for G_2, it is the normal subgroup of G_1, defined by $g = \mathrm{id}$. It is clear that $G_0, G_1, \ldots$ form a decreasing sequence. Using a classical terminology in the theory of Fushsian groups, G_1 is called the group of parabolic transformations in S. A complete description of G_2 will now be given. In general, if L is an arbitrary quasi-hermitian form, it can be expressed uniquely as a sum of a hermitian form and a symmetric form. For instance, the hermitian part H of L is given by

$$H(u, v) = (1/2i)(L(iu, v) - L(u, iv)) .$$

We shall denote the hermitian part of L_t by H_t. This being remarked, if

$$(z, u, t) \longrightarrow (Az + a(u, t), B(t)u + b(t), t)$$

is an arbitrary element of G_2, it decomposes uniquely into a product $(b, a)\gamma$ of (b, a) and γ with γ given explicitly as

$$(z, u, t) \longrightarrow (Az - iAL_t(u, u) + iL_t(B(t)u, B(t)u), B(u)u, t),$$

in which A, $B(t)$, $b(t)$ have the same meaning as in the original element of G_2. We note that necessary and sufficient conditions for a transformation like that of γ to define an analytic automorphism of S are

(1) A <u>is an element of</u> $GL(Z)$ <u>satisfying</u> $AC = C$,

(2) $B : T \longrightarrow GL(U)$ <u>is analytic.</u>

(3) $AH_t(u, u) = H_t(B(t)u, B(t)u)$ <u>for all</u> (u, t) in $U \times T$.

Furthermore, elements like γ form a subgroup $\{\gamma\}$ of G_2, and it is isomorph to the group of pairs (A, B) satisfying the above three conditions. In Pyatetski-Shapiro, the exact form of γ and the crucial condition (3) are stated incorrectly. At any rate, G_3 is the normal subgroup of G_2 defined by A = id., B = id., and we have a semidirect product decomposition $G_2 = G_3 \cdot \{\gamma\}$. Moreover, the law of composition in G_2 is described as

$$((b, a)\gamma)(b', a')((b, a)\gamma)^{-1} = (Bb', Aa' + 4[b, Bb']).$$

2. Using the same notations as in the previous section, we shall introduce a Hausdorff topology in the union $\bar{S} = S \cup T$ so that S becomes an open subset of $\bar{S}$. We have only to assign neighborhoods to each point of T. Let t_0 be a point of T. We take a neighborhood V of t_0 in T and a vector r of R. We then consider the subset $S(V, r)$ of S defined by

$$\mathrm{Im}(z) - \mathrm{Re}(L_t(u, u)) - r \in C, \quad t \in V$$

and take the union $\bar{S}(V, r) = S(V, r) \cup V$ as a neighborhood of t_0 in $\bar{S}$. It is easy to verify that we have a topology in $\bar{S}$ with the required properties. We observe that G_2 operates on $\bar{S}$ as a group of homeomorphisms. In fact we have

$$(b, a)\gamma . S(V, r) = S(V, Ar) .$$

It is also immediate to introduce the structure of a normal ringed space in $\bar{S}$ which induces on S the given structure of the complex analytic manifold.

Now, every element of G_1 gives rise to an analytic automorphism of T as $t \longrightarrow g(t)$. In this way G_1/G_2 can be identified with a subgroup of the group of analytic automorphisms of T. Let Γ_0 be a subgroup of G_0 which is properly discontinuous on S. We shall assume that T is "Γ_0-rational." If we put $\Gamma_k = \Gamma_0 \cap G_k$ for $k = 1, 2, \ldots,$ this means that the quotient space G_3/Γ_3 is compact and that the quotient group Γ_1/Γ_2 is properly discontinuous on T. Since we do not know whether it is a consequence or not, we shall assume, in addition, that if we take V sufficiently small and r sufficiently "large", elements α of Γ_0 with the property $\alpha \cdot S(V, r) \cap S(V, r) \neq \emptyset$ are all contained in Γ_1. We know that this assumption is always satisfied if S is obtained from a bounded symmetric domain and from its boundary component. This being remarked, we take a point t_0 of T which is not a fixed point of Γ_1/Γ_2, and investigate the compactification of the quotient variety of S by Γ_0, which we shall denote simply by S/Γ_0, around the image point of t_0.

In general, let Ω be a discrete subgroup of G_3 such that the quotient space G_3/Ω is compact. It is the same thing to assume that G_3/Ω has a finite volume. We note that the bi-invariant measure in G_3 is the product measure of the ordinary measures in $\mathcal{U}$ and R. At any rate, if Ω is such a group, then $\Omega \cap R$ is discrete in R and the image $\Omega(t)$ of Ω in U is discrete in U for every t both with compact quotient groups. Therefore, if f is an analytic function in $S(V, r)$ and if it is invariant by $\Omega \cap R$, i.e., by the operations of $\Omega \cap R$, it admits a Fourier expansion

$$f(z, u, t) = \sum_{\rho} \theta_{\rho}(u, t)\, e(\rho(z)),$$

in which $\rho : Z \longrightarrow \mathbb{C}$ is $\mathbb{C}$-linear and takes integer values on $\Omega \cap R$. Actually, the series is absolutely and uniformly convergent in every compact subset of $S(V, r)$, and the coefficients define analytic functions on $U \times V$. Furthermore, in case f is invariant by Ω, each θ_{ρ} satisfies the functional equation

$$\theta_{\rho}(u + b(t), t) = e(-\rho(a + iL_t(2u + b(t), b(t))))\theta_{\rho}(u, t)$$

for all (b, a) in Ω. Therefore, for each t in V, the function $u \longrightarrow \theta_{\rho}(u, t)$ is a theta-function (or a "Jacobi function") on U relative to $\Omega(t)$. In particular $\theta_0(u, t)$ depends only on t. The Fourier expansion of $z \longrightarrow f(z, u, t)$ is called the Fourier-Jacobi series of f by Pyatetski-Shapiro. It is easy to determine the Riemann form of θ_{ρ}, more precisely of $u \longrightarrow \theta_{\rho}(u, t)$, in the sense of Weil [12]. In fact, it is simply the hermitian part of the quasi-hermitian form $2i\cdot(-\rho(iL_t(2u, v)))$. Therefore the Riemann form of θ_{ρ} is $4\rho(H_t(u, v))$, and its imaginary part is $4\rho([u, v])$. We note that $4\rho([u, v])$ is integer valued on $\Omega(t) \times \Omega(t)$. In fact ρ takes integer values on $\Omega \cap R$ and, with $b(t)$, $b'(t)$ in $\Omega(t)$, $4[b, b']$ is in $\Omega \cap R$ because of the last formula in Section 1. On the other hand, since the Riemann form has to be positive semi-definite, the summation in the Fourier-Jacobi series of f is restricted by

$$\rho(H_t(u, u)) \geq 0$$

for all u in U. This in general implies that ρ is non-negative on C. On the other hand, if f is bounded in $S(V, r)$ and if $\rho \neq 0$ appears in the Fourier-Jacobi series of f, then ρ is positive on C. (The converse is also true when V is relatively compact with respect to T.) Therefore, by restricting to bounded functions if necessary, we can assume in the following that this condition is satisfied.

Going back to the situation we had before, we apply the above consideration to $\Omega = \Gamma_3$ taking as V an open neighborhood of t_0. Then a formula at the end of Section 1 shows that, if $(b, a)y$ is in Γ_2, A given rise to an automorphism of the lattice Γ_4 and $B(t)$ gives rise to an automorphism of the lattice $\Gamma_3(t)$. Therefore, if we take an affine coordinate system, for instance, in R so that Γ_4 becomes the lattice of integer points, A will be represented by an integer matrix.

3. The general considerations we have made so far will become exceptionally simple if we assume that

(S) <u>the center</u> G_4 <u>of</u> G_3 <u>is one-dimensional.</u>

It is the same thing to assume that R is one-dimensional over $\mathbb{R}$ or Z is one-dimensional over $\mathbb{C}$. We note that, if S is obtained from an irreducible bounded symmetric domain of type I, II, III or IV and from its highest dimensional boundary component, the condition (S) is always satisfied. A. Borel told us that the same is known also for the two exceptional cases of dimensions 16 and 27. This being remarked, if (S) is satisfied, we can identify Z with $\mathbb{C}$ so that R, C, Γ_4 are respectively identified with

$\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, Then the Fourier-Jacobi series will take the following form

$$f(z, u, t) = \sum_{k=0}^{\infty} \theta_k(u, t) e(kz) .$$

Since $\operatorname{Im}(H_t(u, v)) = [u, v]$ is non-degenerate, we have $H_t(u, u) > 0$ for $u \neq 0$. Moreover, the conditions (1) , (3) in Section 1 imply that, if (A, B) comes from an element $(b, a)\gamma$ of Γ_2, we have $A = 1$ and $B(t)$ keeps H_t invariant. Therefore $B(t)$ gives rise to an automorphism of the <u>polarized abelain variety</u> $\mathcal{Q}_t = U/\Gamma_3(t)$, the polarization being determined by the Riemann form $4H_t(u, v)$. In particular $\{B(t)\}$ is a <u>finite group</u> (the structure of which does not depend on t). We shall consider the simplest case assuming that

(Γ) <u>we have</u> $\Gamma_2 = \Gamma_3$.

This means precisely that we have $\{B(t)\} = 1$. In this case, if we take V sufficiently small and r sufficiently large, the quotient space

$$\mathcal{X} = \bar{S}(V, r)/\Gamma_3$$

with the ring of invariant analytic functions on $S(V, r)$ relative to Γ_3 , which is nothing else than the ring of Fourier-Jacobi series, describes the analytic structure of the compactification of S/Γ_0 around t_0 in the sense it gives a neighborhood of the image point of t_0 in the compactification together with the ring of analytic functions on it. This is because t_0 is not a fixed point of Γ_1/Γ_2 . Consider, on the other hand, an open subset W of the product $\mathbb{C} \times U \times V$ with coordinates (w, u, t) satisfying

$$\operatorname{abs}(w) < \exp(-2\pi(\operatorname{Re}(L_t(u, u)) + r)) .$$

where abs(w) means the absolute value of w. Then (b,a) gives rise to an analytic automorphism of W as

$$(w, u, t) \longrightarrow (e(a + iL_t(2u + b(t), b(t)))w, u + b(t), t),$$

and in this way G_3 operates on W. We observe that Γ_4 is precisely the subgroup of G_3 which operates trivially on W, and Γ_3/Γ_4 operates on W properly discontinuously and without fixed points. Hence the quotient variety

$$\mathfrak{X}^* = W/(\Gamma_3/\Gamma_4) = W/\Gamma_3$$

is non-singular. We observe that invariant analytic functions in W are obtained simply by replacing $e(z)$ by w in the Fourier-Jacobi series (of invariant analytic functions in $S(V, r)$ both relative to Γ_3). On the other hand, we note that the closed subvariety $W_0 = \{0\} \times U \times V$ of $\mathbb{C} \times U \times V$ is contained in W. We shall denote its complement in W by W_1. Then both W_0 and W_1 are stable by G_3 and we have $W_1/\Gamma_3 = \mathfrak{X}^* - W_0/\Gamma_3$. More precisely, the quotient variety W_0/Γ_3 is non-singular and it is the closed subvariety of the non-singular variety $\mathfrak{X}^*$ defined by $w = 0$; the quotient variety W_1/Γ_3 is the complement of W_0/Γ_3 in $\mathfrak{X}^*$. Similarly we have $S(V, r)/\Gamma_3 = \mathfrak{X} - V$. Now we shall define a mapping $\mathfrak{X}^* \longrightarrow \mathfrak{X}$. We take a point of $S(V, r)$ with coordinates (z, u, t) and associate the point of W_1 with the coordinates $(e(z), u, t)$. This defines an analytic mapping $S(V, r) \longrightarrow W_1$ and, by passing to quotient varieties, it gives rise to an analytic isomorphism $S(V, r)/\Gamma_3 \simeq W_1/\Gamma_3$. Next we take a point of W_0 with coordinates $(0, u, t)$ and associate the point t of V. This defines an analytic mapping $W_0 \longrightarrow V$ and then an analytic mapping

$W_0/\Gamma_3 \longrightarrow V$, which is surjective and proper. In fact, the fiber over an arbitrary point t of V is the abelian variety $\mathcal{A}_t = U/\Gamma_3(t)$. At any rate, if we combine the two mappings $W_I/\Gamma_3 \longrightarrow S(V,r)/\Gamma_3$ and $W_0/\Gamma_3 \longrightarrow V$, we get a continuous mapping $\mathcal{X}^* \longrightarrow \mathcal{Y}$, which is surjective and proper. The verification is left as an exercise to the reader. We know that this is an analytic isomorphism in the open subset W_1/Γ_3 . Also the remark we made before about the analytic structure of $\mathcal{X}$ around points of V shows that the mapping is analytic around points of W_0/Γ_3 . Therefore $\mathcal{X}^* \longrightarrow \mathcal{X}$ is an analytic mapping or a morphism and the theory of the theta-functions shows that it is a "blowing up" of $\mathcal{X}$ along V . We have thus obtained the following result:

THEOREM 1. Let S be a Siegel domain over T satisfying (S) ; let Γ_0 be a properly discontinuous group of analytic automorphisms of S such that T is Γ_0-rational. Then, if t_0 is not a fixed point of Γ_1/Γ_2 and if Γ_0 satisfies (Γ), a neighborhood of the image point of t_0 in the compactification of S/Γ_0 can be blown up along the image of T to a non-singular variety so that the fiber over the image point of t near t_0 is the abelian variety $\mathcal{A}_t = U/\Gamma_3(t)$.

We note that, in case Γ_0 is not small enough to satisfy (Γ) , we can still blow up the image of T so that the fiber over the image point of t is the generalized Kummer variety $U/\Gamma_2(t)$. This process was investigated by Satake [S] in the case when S is the Siegel upper-half plane and Γ_0 is the Siegel modular group of level 1 . At any rate, Theorem 1 is of fundamental importance because it gives precisely a link between the theory of automorphic functions and the theory of theta-functions.

4. We shall show that the said neighborhood of the image point of t_0 has no finite non-singular coverings. We shall use the same notations as before.

Let ω = dzdudt be the (highest) multiple differential form on the product $Z \times U \times T$ and consider its restriction to $S(V, r)$. Since it is invariant by Γ_3, we get a multiple differential form, which we shall also denote by ω, on the open subset $S(V, r)/\Gamma_3$ of $\mathcal{X}$. We observe that ω is holomorphic at every simple point of $\mathcal{X}$. However, since the (contravariant) image ω^* of ω under the morphism $\mathcal{X}^* \longrightarrow \mathcal{X}$ has the expression $(1/2\pi i)(dw/w)dudt$, this is not holomorphic along $w = 0$. Now, suppose that $\mathcal{X}$ has a finite non-singular covering $\mathcal{Y} \longrightarrow \mathcal{X}$. Then the image $\bar{\omega}$ of ω under $\mathcal{Y} \longrightarrow \mathcal{X}$ is holomorphic everywhere in $\mathcal{Y}$. This depends on the fact that the co-dimension of V in $\mathcal{X}$ hence also the co-dimension of the inverse image of V in $\mathcal{Y}$ are at least two. Consequently, if $\mathcal{Y}^*$ is the Oka normalization [6] of the product $\mathcal{X}^* \times_{\mathcal{X}} \mathcal{Y}$, i.e., if $\mathcal{Y}^*$ is the normalization of the graph of the "mapping" $\mathcal{Y} \longrightarrow \mathcal{X}^*$, the image $\bar{\omega}^*$ of $\bar{\omega}$ under the morphism $\mathcal{Y}^* \longrightarrow \mathcal{Y}$ is holomorphic at every simple point of $\mathcal{Y}^*$. On the other hand, since $\bar{\omega}^*$ is also the image of ω^* under the morphism $\mathcal{Y}^* \longrightarrow \mathcal{X}^*$ and since this is a covering, it is not holomorphic along the inverse image of $w = 0$. This is a contradiction. Therefore $\mathcal{X}$ has no finite non-singular coverings.

This type of argument was suggested to us by Zariski in the special case mentioned in the Introduction. We note that we can arrive at the same conclusion using either the well-known information about the total transform of a simple point or a topological method. We shall formulate our result in the following way:

THEOREM 2. _Let_ S _be a Siegel domain over_ T _satisfying_ (S); _let_ Γ_0 _be a properly discontinuous group of analytic automorphisms of_ S _such that_ T _is_ Γ_0_-rational. Then, if_ Γ_0 _operates on_ S _without fixed points and contains a subgroup_ Γ_0' _of finite index satisfying_ (Γ), _the compactification of_ S/Γ_0 _has no finite non-singular coverings locally at any image point of_ T.

In fact, let t_0 be a point of T which is not a fixed point of Γ_1/Γ_2. Suppose that the compactification of S/Γ_0 has a finite non-singular covering locally at the image point of t_0. Since S/Γ_0' is unramified over S/Γ_0, this covering has to go through the compactification of S/Γ_0'. In this way [cf.5], we get a finite non-singular covering of the compactification of S/Γ_0' locally at the image point of t_0. We know, however, that this is not possible. Since points like t_0 form a dense open subset of T, the compactification of S/Γ_0 has no finite non-singular coverings at any image point of T.

We note that, in case S is obtained from a bounded symmetric domain, the existence of Γ_0' in Theorem 2 can be proved by a method which is formalized by Selberg [9]. We note also that the idea to derive Theorem 2 from Theorem 1 has been suggested to us by M. Artin. In our original formulation, Theorem 2 was stated slightly differently and was proved first (before Theorem 1) as follows. Instead of the assumption that Γ_0 contains a subgroup Γ_0' of finite index satisfying (T), we required that Γ_0 contains a decreasing sequence of subgroups $\Gamma_0^{(n)}$ with certain properties, which is in most cases constructible by Selberg's method, and proved that the dimensions of the Zariski tangent spaces [13] of the compactifications of $S/\Gamma_0^{(n)}$ along the images of T near t_0 tend to infinity with n. This again implies the non-existence of finite non-singular coverings of the compactification of S/Γ_0 at any image point of T. It seems that this crude method can be applied even to the case when (S) is not satisfied.

We note finally that, in some cases, we can dispense with the assumption (T). For instance, in the case of the compactification of the quotient variety of the Siegel upper-half plane of genus g by $Sp(g, \mathbb{Z})$, say, we can blow up the compactification along the boundary so that fibers over general points of the boundary become Kummer varieties of dimension g - 1. Therefore, by a similar argument as before, using $(dzdudt)^2$ instead of dzdudt in the case g is even, we see that the compactification has no finite non-singular coverings locally at any boundary point for $g \geq 3$. The reason why g = 2 is excluded is that the Kummer variety of dimension one is exceptional. Actually, in the case g = 2, we have complete information since we know the structure of the compactification [4, 5]. On the other hand, if we are just interested in whether the boundary is simple or not, i.e., general points of the boundary are simple or not, we estimate the dimension of the Zariski tangent space of the compactification along the boundary. We see easily that this is equal to the number of linearly independent theta-functions of order two and of "characteristic" zero for genus $g-1$, and it is 2^{g-1}. Since the co-dimension of the boundary is g, we get $g = 2^{g-1}$ as a necessary and sufficient condition for the boundary to be simple [cf. 13]. Hence, as it was observed recently by U. Christian [3], the boundary is singular, i.e., all boundary points are singular, for $g \geq 3$ while the boundary is simple for $g = 2$. At any rate, it is understood that, if we take a subgroup of $Sp(g, \mathbb{R})$ commensurable with $Sp(g, \mathbb{Z})$ which operates without fixed points on the Siegel upper-half plane, we can apply Theorem 2 to this subgroup as Γ_0 and we get the non-existence of finite non-singular coverings for all $g \geq 2$.

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