Real

Real Analysis Preliminary Examination, Spring 1994

1. Prove that if $f : [0,1] \to \mathbf{R}$ is a function such that $e^{|f|}$ is integrable, then $f \in L^p([0,1])$ for every $p \geq 1$. Is this conclusion valid if the domain of f is taken to be $\mathbb{R}$?

2. Let V be an infinite-dimensional real Banach space. Prove that any Hamel basis for V (i.e. any basis for V as an abstract vector space over $\mathbf{R}$) must be uncountable.

3. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which is *not* differentiable on some dense set of points.

4. Let X be a measure space with measure μ and $f : X \to \mathbf{R}$ a measurable function.

   a) Show that the set function $f_\bullet\mu$, given by,

   $$(f_\bullet\mu)(E) := \mu(f^{-1}E)$$

   is a Borel measure on $\mathbb{R}$.

   b) Given $g : \mathbb{R} \to \mathbb{R}$, show that g is Borel measurable iff $g \circ f$ is measurable, with

   $$\int_{\mathbb{R}} g\, d(f_\bullet\mu) = \int_X g \circ f\, d\mu$$

   whenever either integral exists.

5. Let μ be any Borel measure on $[0,1]$ with $\mu([0,1]) = 1$. Put $|A|$ for the Lebesgue measure of A. For each $n \in \mathbb{N}$, put E_n to be the set of all $x \in [0,1]$ such that there is an interval $x \in I \subset [0,1]$ with $|I| < 1/n$ and $\mu(I) > 2|I|$. Prove that given any $\epsilon > 0$ we have $|E_n| < \epsilon + 1/2$ for all n sufficiently large.

6. Let X be a compact metric space, and put $K(X)$ for the set of all closed subsets of X. Define the map $\phi : K(X) \to C(X, \mathbb{R})$ by

$$\phi(A)(x) := \mathrm{dist}\,(x, A) := \inf\{\, d(x,y) \mid y \in A \,\}$$

whenever $A \in K(X)$. Prove that ϕ is a one-to-one map and that the image $\phi(K(X))$ is a compact subset of $C(X, \mathbf{R})$.

7. Let $f : \mathbf{R} \to \mathbf{R}$ be absolutely continuous. Prove that

$$\lim_{h \downarrow 0} \int_0^1 \frac{f(x+h) - f(x)}{h}\, dx = f(1) - f(0).$$

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8. Let $C^1([0,1])$ denote the vector space of all continuous functions $f : [0,1] \to \mathbb{R}$ with continuous derivative in $(0,1)$ and for which $\lim_{x\downarrow 0} f'(x)$ and $\lim_{x\uparrow 1} f'(x)$ both exist.

a) Prove that $C^1([0,1])$ is a Banach space under the norm

$$\|f\| := \sup_{x\in[0,1]} |f(x)| + \sup_{x\in(0,1)} |f'(x)|.$$

b) Show that the map $\alpha : C^1([0,1]) \to \mathbb{R} \times C([0,1])$, given by

$$\alpha(f) := (f(0), f'),$$

is a bounded map of Banach spaces, with a bounded inverse.

c) Describe the dual space $C^1([0,1])^*$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Define the functions $g, h : \mathbb{R} \to \mathbb{Z}$ by

$$g(y) := \#\{\, x \in \mathbb{R} \mid f(x-) < y < f(x+) \,\},$$
$$h(y) := \#\{\, x \in \mathbb{R} \mid f(x-) > y > f(x+) \,\}.$$

Prove that g and h are both integrable, with

$$\int_{\mathbb{R}} g(y)\, dy + \int_{\mathbb{R}} h(y)\, dy \leq \mathrm{Var}(f),$$

the total variation of f.