Real Analysis Qualifying Examination

Fall 2023

The five problems on this exam have equal weighting. To receive full credit give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Let $f_n(x) = \dfrac{nx^2}{n^3 + x^3}$.

   (a) Prove that f_n converge uniformly to 0 on $[0, M]$ for any $M > 0$, but does <u>not</u> converge uniformly to 0 on $[0, \infty)$.

   (b) Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ defines a continuous function on $[0, \infty)$.

2. Let $(X, \mathcal{A})$ be a measurable space and μ is a non-negative set function on $\mathcal{A}$ that is finitely additive with $\mu(\emptyset) = 0$. Recall that such a set function is said to be *continuous from below* if

$$\mu\Big(\bigcup_j A_j\Big) = \lim_{j\to\infty} \mu(A_j) \quad \text{whenever } A_j \text{ is an increasing sequence of sets in } \mathcal{A}.$$

   Prove that

$$\mu \text{ is a measure} \quad \Longleftrightarrow \quad \mu \text{ is continuous from below.}$$

3. Prove that

$$1 - \frac{x^2}{2} \le \cos x \le e^{-x^2/2}$$

   for all $|x| \le 1$ and conclude from this that

$$\lim_{n\to\infty} \sqrt{\frac{n}{2\pi}} \int_{|x|\le 1} (\cos x)^n \, dx = 1.$$

   Hint: You may use without proof that $\int_{-\infty}^{\infty} e^{-\pi x^2}\, dx = 1$.

4. Let $a, b > 0$. Prove that

$$\int_{[0,1]\times[0,1]} \frac{1}{x^a + y^b} \, dm_2(x, y) < \infty \quad \Longleftrightarrow \quad \frac{1}{a} + \frac{1}{b} > 1$$

   where m_2 denotes Lebesgue measure on $\mathbb{R}^2$.

   Hint: One possible approach would be to consider separately the regions where $x^a \le y^b$ *and* $x^a > y^b$.

5. Let $f_k \to f$ a.e. on $\mathbb{R}$ with $\sup_k \|f_k\|_{L^2(\mathbb{R})} < \infty$. Prove that $f \in L^2(\mathbb{R})$ and that

$$\lim_{k\to\infty} \int_{\mathbb{R}} f_k g = \int_{\mathbb{R}} fg$$

   for all $g \in L^2(\mathbb{R})$.

   Hint: First consider functions g supported on sets of finite measure and use Egorov's Theorem.

1.

(a)

(i) Fix $M>0$. Since $\left|\frac{nx^2}{n^3+x^3}\right| \leq \frac{x^2}{n^2} \leq \frac{M^2}{n^2}$ $\forall x \in [0,M]$

$$\Rightarrow \sup_{x\in[0,M]} |f_n(x)-0| \leq \frac{M^2}{n^2} \to 0 \text{ as } n\to\infty$$

(ii) Since $\sup_{x\in[0,\infty)} \left|\frac{nx^2}{n^3+x^3}\right| \geq \frac{n^3}{n^3+n^3} = \frac{1}{2}$

take $x=n$

$$\Rightarrow \sup_{x\in[0,\infty)} |f_n(x)-0| \not\to 0 \text{ as } n\to\infty.$$

(b) In light of (ii) above we know $\sum_{n=1}^{\infty} f_n$ does <u>not</u> conv unif on $[0,\infty)$, but it <u>does</u> conv. unif on $[0,M]$ $\forall M>0$.

[Since $\left|\frac{nx^2}{n^3+x^3}\right| \leq \frac{M^2}{n^2}$ $\forall x\in[0,M]$ & $\sum_{n=1}^{\infty} \frac{M^2}{n^2} < \infty$

$\underset{\text{M-test}}{\Rightarrow}$ $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ conv. unif on $[0,M]$.]

Thus, $\sum_{n=1}^{\infty} f_n(x)$ defines a continuous function on $[0,M]$ for any $M>0$ & hence a conts fn on $[0,\infty)$.

2.

$\Leftarrow$: Let $E_1, E_2, \ldots$ be a countable collection of disjoint subsets of $\mathcal{A}$. Let $A_k = \bigcup_{j=1}^{k} E_k$, then $A_k \subseteq A_{k+1}$.

$$\Rightarrow \mu\left(\bigcup_{j=1}^{\infty} E_j\right) = \mu\left(\bigcup_{k=1}^{\infty} A_k\right)$$

$$= \lim_{k \to \infty} \mu(A_k)$$

finite add. $\nearrow$

$$= \lim_{k \to \infty} \sum_{j=1}^{k} \mu(E_j)$$

$$= \sum_{j=1}^{\infty} \mu(E_j).$$

$\Rightarrow$: Let $A_1 \subseteq A_2 \subseteq \cdots$ be increasing sequence of sets in $\mathcal{A}$.

Set $A_0 = \phi$ & $E_j = A_j \setminus A_{j-1}$, then $\bigcup_j E_j = \bigcup_j A_j$ &

$$\mu\left(\bigcup_{j=1}^{\infty} A_j\right) = \sum_{j=1}^{\infty} \mu(E_j) = \lim_{n \to \infty} \sum_{j=1}^{n} \mu(E_j)$$

$$= \lim_{n \to \infty} \underbrace{\mu\left(\bigcup_{j=1}^{n} E_j\right)}_{= \mu(A_n)}$$

3.

(a) Since

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \& \quad e^{-x^2/2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}$$

and both $\frac{x^{2n}}{(2n)!}$ & $\frac{x^{2n}}{2^n n!}$ are decreasing when $|x| \leq 1$

it follows that

(i) $1 - \frac{x^2}{2} \leq \cos x \leq 1 - \frac{x^2}{2} + \frac{x^4}{24}$

& (ii) $e^{-x^2/2} \geq 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} \geq 1 - \frac{x^2}{2} + \underbrace{\frac{x^4}{8} - \frac{x^4}{12}}_{=\frac{x^4}{24}}$

provided $|x| \leq 2$.

$x = \sqrt{\frac{2\pi}{n}}\, y$

(b)

$$\underbrace{\sqrt{\frac{n}{2\pi}} \int_{-1}^{1} \left(1 - \frac{x^2}{2}\right)^n dx}_{= \int_{|y| \leq \sqrt{\frac{n}{2\pi}}} \left(1 - \frac{\pi y^2}{n}\right)^n dy} \leq \sqrt{\frac{n}{2\pi}} \int_{-1}^{1} (\cos x)^n dx \leq \underbrace{\sqrt{\frac{n}{2\pi}} \int_{-1}^{1} e^{-nx^2/2} dx}_{= \int_{|y| \leq \sqrt{\frac{n}{2\pi}}} e^{-\pi y^2} dy}$$

$$\leq \int_{-\infty}^{\infty} e^{-\pi y^2} dy = 1$$

$$\longrightarrow \int_{-\infty}^{\infty} e^{-\pi y^2} dy = 1$$

By DCT since $\chi_{|y| \leq \sqrt{\frac{n}{2\pi}}}(y) \left(1 - \frac{\pi y^2}{n}\right)^n \leq e^{-\pi y^2} \ \forall n$ in $L^1(\mathbb{R})$.

(*) consequence of (a).

4.

Soln 1

Let $E = \{(x,y) \in [0,1]\times[0,1] : x^a + y^b < 1\}$

Note that if $\delta E := \{(\delta^{1/a}x, \delta^{1/b}y) : (x,y)\in E\}$, then $m(\delta E) = \delta^{\frac{1}{a}+\frac{1}{b}} m(E)$

$$\int_E \frac{1}{x^a+y^b} = \int_1^\infty \underbrace{m\left(\left\{(x,y) : \frac{1}{x^a+y^b} > \lambda\right\}\right)}_{= m(\lambda^{-1}E)} d\lambda = m(E)\int_1^\infty \lambda^{-\left(\frac{1}{a}+\frac{1}{b}\right)} d\lambda$$

$< \infty$ iff $\frac{1}{a}+\frac{1}{b} > 1$

Soln 2

Write $[0,1]\times[0,1] = A \sqcup B$ with $A = \{x^a \geq y^b\}$ & $B = \{x^a < y^b\}$.

On A we have that $\frac{1}{x^a+y^b} \approx \frac{1}{x^a}$, while on B we have $\frac{1}{x^a+y^b} \approx \frac{1}{y^b}$.

Thus $\int_{[0,1]\times[0,1]} \frac{1}{x^a+y^b} dm_2 < \infty \iff \int_A \frac{1}{x^a} dm_2 < \infty$ & $\int_B \frac{1}{y^b} dm_2 < \infty$

Now $\int_A \frac{1}{x^a} dm_2 \underset{\substack{\uparrow \\ \text{Tonelli}}}{=} \int_0^1 \int_0^{x^{a/b}} \frac{1}{x^a} dy\, dx = \int_0^1 x^{\frac{a}{b}-a} dx < \infty$

iff $\frac{a}{b} - a > -1$

$\Updownarrow$

$\frac{1}{a} + \frac{1}{b} > 1$.

Argument for $\int_B \frac{1}{y^b} dm_2$ is the same.

5.

$M = \sup_k \|f_k\|_2$

(a) $f_k \to f$ a.e $\Rightarrow |f_k|^2 \to |f|^2$ a.e.

Fatou $\Rightarrow \displaystyle\int |f|^2 = \int \liminf |f_k|^2 \le \liminf \int |f_k|^2 \le M^2.$

(b) Suppose $\operatorname{supp}(g) \subseteq E$ with $m(E) < \infty$. Let $\varepsilon > 0$

Since $|g|^2 \in L^1$ $\exists\, \delta > 0$ s.t $\displaystyle\int_A |g|^2 < \varepsilon^2$ whenever $m(A) < \delta$.

Egorov $\Rightarrow \exists\, F$ with $m(E \setminus F) < \delta$ s.t $f_k \to f$ unif on F.

Hence $\exists\, K$ s.t. if $k \ge K$, then $|f_k(x) - f(x)| < \varepsilon$ $\forall x \in F$, thus

$$\left| \int_E (f_k - f) g \right| \le \underbrace{\int_F |f_k - f|\,|g|}_{} + \underbrace{\int_{E\setminus F} |f_k - f|\,|g|}_{}$$

$$\le \varepsilon \int_F |g| \qquad\qquad \le \|f_k - f\|_2 \left(\int_{E \setminus F} |g|^2 \right)^{1/2}$$

Cauchy-Schwarz

$$\le \varepsilon\, m(F)^{1/2} \|g\|_2 \qquad\qquad \le 2M\varepsilon$$

$$\le \varepsilon \left(m(F)^{1/2} \|g\|_2 + 2M \right).$$

Now suppose $m(E) = \infty$. Apply the result above gives

$$\lim_{k \to \infty} \int (f_k - f) g_N = 0 \quad \forall N \text{ with } g_N = g\chi_{|x| \le N}$$

Since $g \in L^2$ $\exists\, N$ s.t $\displaystyle\int_{|x| \ge N} |g|^2 < \varepsilon^2$

Cauchy-Schwarz.

$$\Rightarrow \left| \int (f_k - f) g \right| \le \underbrace{\int |f_k - f|\,|g_N|}_{<\,\varepsilon \ \forall N \text{ (provided } k \text{ large enough)}} + \underbrace{\int |f_k - f|\,|g - g_N|}_{\le\, \|f_k - f\|_2 \|g - g_N\|_2 \,\le\, 2M\varepsilon}.$$