# COMPLEX ANALYSIS EXAM – FALL 2025

*All problems are of equal weight. Please arrange your solutions in numerical order even if you do not solve them in that order. Show work and carefully justify/prove your assertions.*

Problem 1. (20 points) For $a > 0$, compute:

$$\int_0^\infty \frac{x \sin x}{x^2 + a^2}\, dx$$

using contour integration.

Problem 2. (20 points) Let $R, M$ be positive constants and let $\mathbb{D}_R, \mathbb{D}_M$ denote the open discs of radii $R$ and $M$, respectively. Let $f : \mathbb{D}_R \to \mathbb{D}_M$ be holomorphic with $f(0) = 0$.

(a) Prove that $|f(z)| \leq \frac{M|z|}{R}$ for all $z \in \mathbb{D}_R$.

(b) Suppose that $|f'(0)| = \frac{M}{R}$. Show that $f(z)$ is a composition of dilation and rotation.

(c) Suppose $f$ has two distinct fixed points in $\mathbb{D}_R$. Prove that $f(z) = \frac{Mz}{R}$.

Problem 3. (20 points)

(a) Let $\alpha$ be a complex number of modulus $|\alpha| > e = \exp(1)$. How many solutions the equation

$$\alpha z \exp(z) = 1$$

has in $\mathbb{D}$ (the open unit disc)?

(b) Let $f$ be meromorphic in $\overline{\mathbb{D}}$ with no zeros/poles on $\partial\mathbb{D}$. If $|f(z)| = 1$ on $\partial\mathbb{D}$, show that the number of zeros equals the number of poles inside $\mathbb{D}$.

Problem 4. (20 points) Let $G = \mathbb{C} \setminus \{(-\infty, -1] \cup [1, \infty)\}$. Find a conformal map from $G$ to the upper half-plane $\mathbb{H}$.

Problem 5. (20 points) Let $F(z)$ be holomorphic near $z_0$ with $F(z_0) = F'(z_0) = 0$ and $F''(z_0) \neq 0$.

(a) Show that there exist two curves $\Gamma_1, \Gamma_2$ intersecting orthogonally at $z_0$ where $F$ is real-valued.

(b) Prove that $F$ has a saddle point at $z_0$.

Problem 6. (20 points) Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence $r$, with a simple pole at $z_0$ where $|z_0| = r$.

(a) Prove that $a_n \neq 0$ for all sufficiently large $n$.

(b) Show that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$.

Problem 7. (20 points) Let $\{f_k(z)\}, k = 1, 2, 3, \ldots,$ be a sequence of functions holomorphic in $\mathbb{D}$ (the open unit disc). Suppose that

$$F(z) = \sum_{k=1}^{\infty} f_k(z)$$

is uniformly convergent in $\mathbb{D}$. Prove that $F(z)$ is holomorphic in $\mathbb{D}$.

*Good Luck!*