99 PROBLEMS

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Legend: (*) harder; (**) even harder

1. SOME ALGEBRA

(1) Assuming $h \neq 0$, what is \( \frac{f(x+h)-f(x)}{h} \) where \( f(x) = (x+1)^2 \)? Simplify.

(2) Find the domain of the function

\[
f(x) = \frac{\sqrt{x+2} + \log_2(5-x)}{x}.
\]

(3) (*) Consider the function \( f(x) = \ln \left( x + \sqrt{1 + x^2} \right) \). Find the domain of \( f \). Determine the parity of this function, i.e. is it odd, even, or neither?

(4) What is the equation of the secant line joining the points of the graph \( f(x) = 2^x \) whose \( x \)-coordinates are respectively 1 and 2?

(5) Find the point(s) of intersection of the hyperbolas \( x^2 + 3xy = 54 \) and \( xy + 4y^2 = 115 \).

2. LIMITS

Finding the limit at a real value without using l’Hôpital’s rule

\[
\begin{align*}
(6) \lim_{x \to 3} x^2 - 7x + 12 + \sqrt{x^2 - 5} &= & (9) \lim_{x \to 0} \frac{x^4 + 5x - 3}{2 - \sqrt{x^2 + 4}} = \\
(7) \lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 - 5x + 6} &= & (10) \lim_{x \to 1} \frac{x^3 - 1}{(x - 1)^2} = \\
(8) \lim_{x \to 4} \frac{3 - \sqrt{x + 5}}{x - 4} &= & (11) (*) \lim_{x \to 0} x^4 \cos\left(\frac{2}{x}\right) = 
\end{align*}
\]

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Limits of trigonometric type

\( \lim_{x \to 0} \frac{\sin^2 5x}{2x \tan 3x} = \) \hspace{1cm} \( \lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1} = \)

Limits at infinity

\( \lim_{x \to -\infty} \frac{7}{x^3 - 4} = \) \hspace{1cm} \( \lim_{x \to -\infty} \frac{7x^2 - x + 11}{4 - x} = \)
\( \lim_{x \to \infty} \frac{10}{x^2 + 10} = \) \hspace{1cm} \( \lim_{x \to \infty} \frac{\sqrt{9x^2 - 5x}}{x} = \)
\( \lim_{x \to \infty} \frac{7x^2 + x - 100}{2x^2 - 5x} = \) \hspace{1cm} \( \lim_{x \to \infty} \left( \frac{x - 2}{x - 1} \right)^x = \)
\( \lim_{x \to \infty} x^2 - \sqrt{x^2 + 7} = \)

One sided limits

\( \lim_{x \to 3^+} \frac{x^2 + 3x}{9 - x^2} = \) \hspace{1cm} \( \lim_{x \to 3^-} \frac{x^2 - 3x}{x^2 - 9} = \)

3. Asymptotes

(23) The line \( y = mx + p, \) with \( m \neq 0 \) is an oblique asymptote (or slant asymptote) of \( f(x) \) iff \( \lim_{x \to \infty} \frac{f(x)}{x} = m \) and \( \lim_{x \to \infty} f(x) - mx = p. \) Show that \( f(x) = \sqrt{x^2 - 4x} \) has an oblique asymptote at \( \infty \) and a different one at \( -\infty. \)

(24) (*) Show that if \( f(x) \) is a rational function then \( f(x) \) has an oblique asymptote iff the degree of the numerator is exactly one more than the degree of the denominator. [hint: how can you write \( f(x) \) after performing polynomial division?]. Find the oblique asymptote(s) of \( f(x) = \frac{x^2 - 6x + 1}{x - 2} \) using (a) the above definition and (b) using long division.

(25) (*) Can a rational function have two distinct oblique asymptotes?
Find all asymptotes (vertical, horizontal and/or oblique) of the following functions

26) \( e(x) = \frac{x^2 - 4x}{2x + 1} \)
27) \( f(x) = \frac{2x + 1}{3x + 2} \)
28) \( h(x) = \frac{x^4 + 1}{x^2 - 1} \)
29) \( i(x) = \frac{x^3}{x^2 + 1} \)
30) \( j(x) = 2x - \sqrt{4x^2 + 4} \)

31) Find all the asymptotes (if any) to the function \( f(x) = \frac{x^2 - 1}{x|x + 1|} \)

32) (*) Consider the function \( f(x) = ax - \sqrt{bx^2 - 1} \) where \( b \geq 0 \). For which value(s) of \( a \) and \( b \) does this function have an oblique asymptote of slope 5 at \(-\infty\) and of slope 1 at \( +\infty \)?

4. DERIVATIVES

33) Using the limit definition, compute \( f'(3) \) where \( f(x) = x^2 + \frac{2}{x} \)

Compute the derivatives of the following functions:

34) \( f(x) = 4x^5 - 5x^4 \)
35) \( g(x) = 3x^2(x^3 + 1)^7 \)
36) \( h(x) = \frac{(3x - 1)^2}{x^2 + 2x} \)
37) \( i(x) = \frac{-x^2}{\sqrt{1 - \ln^2(x)}} \)
38) \( j(x) = (\arctan(2x))^{10} \)
39) \( k(x) = x^7(x^2 - x)^5 \sin^4(x^2)e^{4x} \)
40) \( l(x) = \arcsin(2\sin(x)) \)
41) \( m(x) = \log_5(3x^2 + x) \)
42) \( n(x) = \frac{3\sin(x) + 2}{4\sin(x) + 3} \)

43) Determine the following limit quickly: \( \lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x - 2} \).
44) Find \( f'(\frac{3\pi}{2}) \) where \( f(x) = (\cos x + 1)^x \).
45) If \( f(2) = 3 \), \( g(2) = 4 \), \( g(3) = 2 \), \( f'(2) = 5 \) and \( g'(3) = 2 \) find
\[
\left( \frac{f(g(x)) + x}{f^2(2x - 4)} \right)'
\]
at \( x = 3 \).
46) Find \( \frac{dy}{dx} \) where \( y \) is a differentiable function satisfying \( \frac{\sin y}{y^2 + 1} = 3x \).

5. TANGENTS

47) Find the point of intersection of the lines tangent to the graph of \( f(x) = x \sin(x) \) at \( x = \frac{\pi}{2} \) and \( x = \pi \).
(48) Find the tangent(s) to the graph of \( f(x) = x^2 - 2x + 1 \) passing through the point (4, 1).

(49) Find the equation of the line tangent at (1, 1) to the graph of the function
\[
y^3 + xy = x^3 - x + 2.
\]

(50) (*) (Legendre Transform) Consider a smooth convex function \( f(x) \). Pick a slope \( m \) and let \( f^*(m) \) be the y-intercept of the tangent to the graph of \( f(x) \) whose slope is \( m \). Find the function \( f^*(m) \) where \( f(x) = x^2 - 2x \).

6. Extrema & Concavity

(51) The function \( f(x) = a \ln x - a^3 x \) has a local minimum at \( x = 4 \) for \( a \neq 0 \). What is \( a \)?

(52) Over which interval is \( f(x) = x^3 - 6x^2 + 3x \) (a) concave up? (b) decreasing?

7. Study of Functions

Study the following functions. I.e. find the (1) domain, (2) asymptotes and/or discontinuities, study the (3) growth and (4) concavity; locate (5) all extrema and inflection points; (6) find the roots and (7) sketch the graph

\[
\begin{align*}
(53) & \quad x^3 - 3x^2 \\
(54) & \quad x^4 - 2x^3 \\
(55) & \quad \frac{3x + 4}{2x + 3} \\
(56) & \quad \frac{x^4 - 4}{x^3} \\
(57) & \quad \frac{x(x - 3)^2}{(x - 2)^2} \\
(58) & \quad \sqrt{1 - x^2} \\
(59) & \quad x^2 - 1 \\
(60) & \quad 3\left( \sqrt{x^2 + 1} - 1 \right) \\
(61) & \quad \frac{1}{x} - \frac{1}{x(1 - x)} \\
(62) & \quad \frac{x - 3}{x - 2} - \frac{3}{x - 2} \\
(63) & \quad \frac{x}{x - 2} - 3 \\
(64) & \quad (*) \, 3x^2 - 2x \\
(65) & \quad \sin(2x) - 2x \\
(66) & \quad (*) \, \text{Consider the function } f(x) = \frac{1}{x^2/3x^2 + 2}. \text{ Study and sketch the function. Using the previous graph, plot (a) } \phi(x) = e^{f(x)} \text{ and, (b) } \psi(x) = f(|x|).
\end{align*}
\]

8. Varia

(67) Give a lower bound on the number of roots of \( f(x) = \cos(\pi x)/x \) on the interval [1, 3]. [hint: Intermediate value theorem]

(68) Suppose that a function \( f(x) \) has a maximum at \( x = 3 \). True or False? Justify.
- The function \( f^2(x) \) has a maximum at 3.
- The function \( e^{f(x)} \) has a maximum at 3.
- The function \( f(x - 3) \) has a maximum at 0.
(69) Without a calculator estimate \( \sin^2 \left( \frac{0.99\pi}{4} \right) \).

(70) If \( -1 \leq f'(x) \leq 3 \) for all \( x \) in \([1, 4]\) and \( f(2) = 4 \), find the maximal and minimal possible values of \( f(4) \).

(71) (**) Suppose that \( f : [0, 1] \to [0, 1] \) is a continuous function. Prove that \( f \) has a fixed point in \([0, 1]\), i.e., there is at least one real number \( x \) in \([0, 1]\) such that \( f(x) = x \).

(72) (**) Suppose that \( g \) is a continuous function on \([0, 2]\) satisfying \( f(0) = f(2) \). Show that there is at least one real number \( x \) in \([1, 2]\) with \( f(x) = f(x - 1) \).

(73) Suppose that \( \sum_{i=1}^{10} a_i = 100 \) compute \( \sum_{i=1}^{10} (2a_i + 3 - i) \).

9. Related Rates

(74) A 10ft ladder is leaning against the wall. How fast is the bottom of the ladder sliding when the top part is 3ft above the ground and gliding at a rate of 1ft per second.

(75) A conical cup has a diameter of 4cm and a height of 8cm. How fast is the level dropping when the height is 4cm and the water escapes from the bottom at a rate of 1cm\(^3\)

10. Optimization

(76) Find the maximal area of rectangle whose sides are parallel to the coordinate axes and whose vertices lie on the curve of equation \( x^2 + y^4 = 1 \).

(77) We have 12 m\(^2\) of material to make a box whose bottom is square and sides are rectangular (the box has no top). What is the maximal volume that such a box can have?

11. Integration

(78) Using 4 rectangles and the right endpoint method estimate \( \int_{0}^{12} \frac{2}{x + 2} \, dx \).

(79) Compute the area under the graph of \( g(x) = x + 3x^3 - \sin(2x) + xe^{x^2} + x^2 \) over the interval \([-3, 3]\).

Compute the following integrals

\[
\begin{align*}
(80) & \int_{0}^{1} xe^{-x^2} \, dx \\
(81) & \int_{-2}^{3} |x - 1| \, dx \\
(82) & \int_{0}^{1} (\sin x + \cos x)^2 \, dx \\
(83) & \int_{-2}^{3} \frac{x^3}{x^2 + \pi} \, dx \\
(84) & \int_{0}^{1} \frac{x^4 - 3x^2}{x^2} \, dx \\
(85) & \int_{0}^{5} 2x \, dx \\
\end{align*}
\]
99 PROBLEMS

(86) \((*) \int_0^1 \frac{x}{\sqrt{x+1}} \, dx\)

(87) \((*) \int \frac{1}{1+e^x} \, dx\)

(88) \((**) \int \frac{1}{1+\sin^2(x)} \, dx\)

(89) Compute the area of the region between the parabolas \(y = 2x^2 - 2\) and \(y = x^2 + x\).

(90) Compute the area of the region bound by \(y = x^3 + x\), \(y = x^3\), \(x = -2\) and \(x = 1\).

12. FUNDAMENTAL THEOREM OF CALCULUS

(91) Find \(f'(x)\) where \(f(x) = \int_x^{x^2} \frac{\sin t}{t} \, dt\).

13. GRAPH ANALYSIS

Based on the above picture representing the graph of \(f(x)\), answer the following questions.

(92) \(\lim_{x \to 1^+} f(x) = \)

(93) \(\lim_{x \to 3} f(x) = \)

(94) \((*) \lim_{x \to -2^+} f(-x) = \)

(95) \(f'(\frac{3}{2}) = \)

(96) \(\int_3^4 f(x) \, dx\)

(97) \(F'(4) = \) where \(F(x) = \int_0^x f(t) \, dt\)

(98) Sketch \(f'(x)\)

(99) Sketch \(F(x) = \int_1^x f(t) \, dt\)

\diamond\diamond\diamond