UNIVERSITY OF GEORGIA

DEPARTMENT OF MATHEMATICS
MATH 2250 - FINAL EXAM
FALL 2025

PRINTED FIRST & LAST NAME

— SOLUTIONS —

UGAID – YOUR 81#

INSTRUCTOR

CLASS TIME

INSTRUCTIONS

This exam has **20** questions and is out of **140** points.

- The exam lasts 3 hours and it has two parts: the first one consists of Multiple Choice (MC) questions, and the second part of Free Response (FR) ones. You must show work for both parts, unless explicitly told otherwise. An unjustified answer will receive no credit. If you are using a shortcut, explain it.
- Your work must be neat and organized. For multiple-choice questions, **bubble in** your answer, and for free-response questions, enter your answer in the provided box. There is only one correct choice for each MC question.
- Smart devices, including smart watches and cell phones, are prohibited and must not be within reach.
- If you plan to use a calculator, only TI-30XS MultiView (the name must match exactly) is permitted; no other calculators or sharing of calculators is allowed.
- Provide an exact answer for each problem. Answers containing symbolic expressions such as cos(3) and ln(2) are perfectly acceptable, but please simplify or evaluate when possible, for example, $\sin(\pi/2) = 1$.
- If additional space is needed, use the last two pages. Write "cont'd" (continued) in the designated area and continue on the scrap paper by first writing the problem number and then continuing your solution. Work outside the specified area without any indication will not be graded.

SCRAP PAPER

Do NOT tear this page off!

Part I: Multiple Choice

Show work and **bubble in** the final answer.

1. [5 pts] If $y = x^{\pi} + \sin(\pi x) + \pi x$, find $y'(1)$.

- (A) $\pi - 1$
- (B) π
- (C) 2π
- (D) 3π
- (E) Some other answer.

$$y'(x) = \pi x^{\pi-1} + \pi \cos(\pi x) + \pi$$

$$y'(1) = \pi \cdot 1^{\pi-1} + \pi \cos(\pi) + \pi$$

$$= \pi - \pi + \pi$$

$$= \boxed{\pi}$$

2. [5 pts] Consider the function $f(x) = x - \ln(x)$ on $[1, e]$. This function has an absolute maximum at the point (a, b). What is $b - a$?

- (A) -1
- (B) 0
- (C) 1
- (D) 2
- (E) $e - 2$

$f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$: critical value $x = 1$

$f(1) = 1 - \ln 1 = 1$

$f(e) = e - \ln e = e - 1$

$f(x)$ has an absolute maximum at $(e, e-1)$ where $a = e$ and $b = e-1$.

So $b - a = e - 1 - e = \boxed{-1}$.

3. [5 pts] Let $f(x)$ and $g(x)$ be differentiable functions with the following values and derivatives at specific points:

	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
$x=0$	4	1	-1	3
$x=1$	3	2	-2	0

If $h(x) = \dfrac{f(x)+x}{g(x)}$, find $h'(1)$.

- (A) -1
- (B) $-\frac{1}{2}$
- (C) 0
- (D) 1
- (E) $\frac{1}{2}$

$$h'(x) = \frac{g(x)\left[f'(x)+1\right] - g'(x)\left[f(x)+x\right]}{[g(x)]^2}$$

$$h'(1) = \frac{\underbrace{g(1)}_{=2}\left[\underbrace{f'(1)}_{=-2}+1\right] - \underbrace{g'(1)}_{=0}\left[\underbrace{f(1)}_{=3}+1\right]}{[\underbrace{g(1)}_{=2}]^2}$$

$$= \frac{-2}{4}$$

$$= \boxed{-\frac{1}{2}}$$

4. [5 pts] Let $f(t)$ be a function defined over the interval $(-1, \infty)$. Find $f'(0)$ given that

$$f(t) = \ln\left(2^{\arctan(t)}(1+t)\right).$$

- (A) -1
- (B) 0
- (C) $\ln(2)$
- (D) $\ln(2) - 1$
- (E) $\ln(2) + 1$

Using the properties of logarithms:

$$f(t) = \arctan t \cdot \ln 2 + \ln(1+t)$$

$$f'(t) = \frac{\ln 2}{1+t^2} + \frac{1}{1+t}$$

$$f'(0) = \frac{\ln 2}{1+0^2} + \frac{1}{1+0}$$

$$= \boxed{\ln 2 + 1}$$

5. [5 pts] The surface area of an inflatable cubical balloon is increasing at a rate of 6 cm^2 per minute. How fast is its side length increasing when the volume is 27 cm^3?

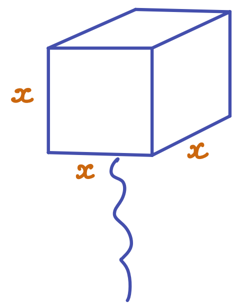
(A) $\frac{1}{18}$ cm/min

(B) $\frac{1}{6}$ cm/min

(C) $\frac{1}{3}$ cm/min

(D) 2 cm/min

(E) 3 cm/min



Given: $\frac{dS}{dt} = 6 \ \text{cm}^2/\text{min}$

Wanted: $\frac{dx}{dt} = ?$ when $V = 27 \text{ cm}^3$

i.e. $V = x^3 = 27$

$x = 3$ cm

Relation: $S = 6x^2$

$\frac{dS}{dt} = 12x \frac{dx}{dt}$ ($\frac{d}{dt}$)

Substitute: $6 = 12 \cdot 3 \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{\cancel{6}}{\underset{2}{\cancel{12}} \cdot 3} = \frac{1}{6}$

The side length of the balloon is $\boxed{\text{increasing at a rate of } \frac{1}{6} \frac{\text{cm}}{\text{min}}}$.

6. [5 pts] Let $f(x)$ be a function that is defined and differentiable for all real numbers, and assume that $f(x)$ is increasing only over the interval $(1,3)$. Where is the function $e^{f(x)}$ necessarily increasing?

(A) Over the interval (e, e^3).

(B) Over the interval $(1,3)$.

(C) Over the intervals $(-3,-1)$ and $(1,3)$.

(D) Everywhere as e^x is strictly increasing.

(E) We can't answer as we do not have enough information about f.

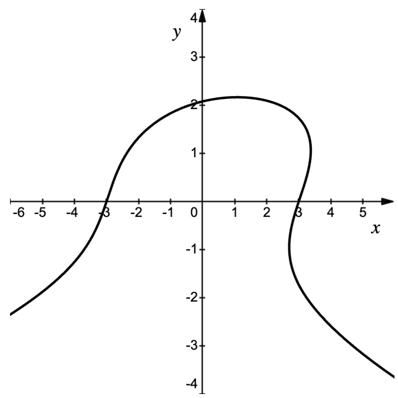
$\left(e^{f(x)}\right)' = f'(x) \underbrace{e^{f(x)}}_{>0} > 0$ when $f'(x) > 0$ and $f'(x) > 0$ on $(1,3)$

so $\boxed{e^{f(x)} \text{ is increasing on } (1,3)}$.

7. [5 pts] Find the equation of the line tangent to the graph of

$$x^2 + y^3 - xy = 9$$

at the point $(3, 0)$.



Ⓐ $y = \frac{1}{2}x - \frac{3}{2}$

Ⓑ $y = x - 3$

Ⓒ $y = 2x - 6$

Ⓓ $y = 2x - 9$

Ⓔ $y = 3x - 9$

Differentiating both sides of the equation with respect to x:

$2x + 3y^2y' - xy' - y = 0$

At the point $(3,0)$, the above equation reduces to:

$6 - 3y' = 0 \Leftrightarrow y' = 2$

Equation of the tangent line at $(3,0)$:

$y - 0 = 2(x-3) \Leftrightarrow \boxed{y = 2x - 6}$

8. [5 pts] Determine the types of discontinuities of

$$f(x) = \frac{x(x^2 - 4)}{(x^2 - 5x + 6)e^x}.$$

Ⓐ The function has infinite discontinuities at $x = 2$ and $x = 3$.

Ⓑ The function has a removable discontinuity at $x = 2$ and an infinite discontinuity at $x = 3$.

Ⓒ The function has a removable discontinuity at $x = 3$ and an infinite discontinuity at $x = 2$.

Ⓓ The function has infinite discontinuities at $x = -2$ and $x = 3$.

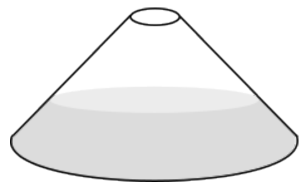
Ⓔ The function has removable discontinuities at $x = 0$, $x = 2$, and $x = 3$.

$f(x) = \dfrac{x(x-2)(x+2)}{(x-2)(x-3)e^x}$ $f(x)$ is discontinuous at $x = 2$ and $x = 3$.

$\lim\limits_{x\to 2} \dfrac{x\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-3)\underbrace{e^x}_{>0}} = \dfrac{4}{-e^2}$, so $f(x)$ has a $\boxed{\text{removable discontinuity at } x = 2}$.

$\lim\limits_{x\to 3^+} \dfrac{\overset{+}{x}\overset{+}{(x-2)}\overset{+}{(x+2)}}{\underset{+}{(x-2)}\underset{+}{(x-3)}\underset{+}{e^x}} \left[\frac{15}{0}\right] = +\infty$, so $f(x)$ has an $\boxed{\text{infinite discontinuity at } x = 3}$.

9. [5 pts] A conical bottle (as depicted below) gets filled at a constant rate. What can you say about the graph of $h(t)$, the function giving the height in terms of time, t?
Hint: You do not need to make any computations to solve this problem; think of it concretely.



- (A) It is increasing concave up.
- (B) It is increasing concave down.
- (C) It is decreasing concave up.
- (D) It is decreasing concave down.
- (E) It is increasing linearly.

$h(t)$ is increasing as you fill it.
As the water level gets higher and higher, the bottle is getting narrower, hence it fills up faster.
So $h(t)$ is increasing at an increasing rate, i.e. it is increasing concave up.

10. [5 pts] Consider the family of piecewise functions

$$f(x) = \begin{cases} 3x - 2, & x < 1, \\ kx^3 + k^2 - 1, & x \geq 1, \end{cases}$$

depending on a parameter k. For what value(s) of k is f differentiable at $x = 1$?

- (A) $k = -2$
- (B) $k = -1$ and $k = 1$
- (C) $k = -2$ and $k = 1$
- (D) $k = 1$
- (E) $k = 0$ and $k = 1$

• For $f(x)$ to be differentiable at $x=1$, it must be continuous at $x=1$:

$$\lim_{x\to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$3-2 = \underbrace{k + k^2 - 1}$$

$$\iff k^2 + k - 2 = 0$$

$$\iff (k-1)(k+2) = 0 \Rightarrow k=-2,\ k=1.$$

• $f'(x) = \begin{cases} 3 & x<1 \\ 3kx^2 & x \geq 1 \end{cases}$

For $f(x)$ to be differentiable at $x=1$:

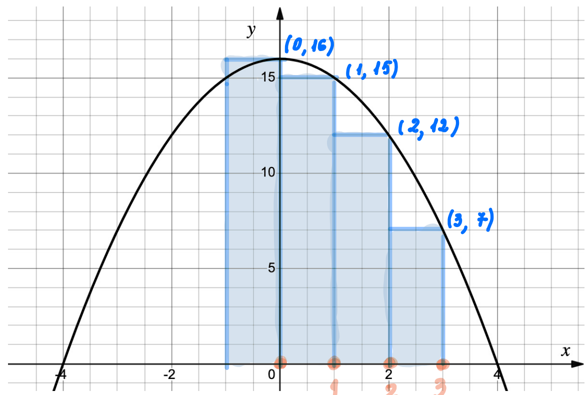
$$\lim_{x\to 1} f'(x) = \lim_{x\to 1^+} f'(x)$$

$$3 = 3k \Rightarrow k = 1.$$

$f(x)$ is differentiable at $x=1$ when $\boxed{k=1}$.

11. [5 pts] Estimate the area under the graph of $f(x) = 16 - x^2$ over the interval $[-1, 3]$. Use a Right Riemann sum with four subintervals of equal width.

(A) 25
(B) 48
(C) 50
(D) 58
(E) 68



$$\Delta x = \frac{3-(-1)}{4} = 1$$

x	$f(x)$
0	16
1	15
2	12
3	7

$$R_4 = \sum_{i=1}^{4} f(x_i)\,\Delta x = [16 + 15 + 12 + 7] \cdot 1$$
$$= 50$$

$$\boxed{A \approx R_4 = 50}$$

12. [5 pts] Compute $\displaystyle\int_0^8 \sin\left(\frac{\pi x}{16}\right) dx$.

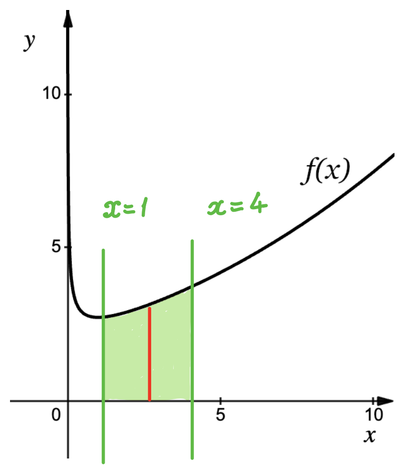
(A) $-\frac{16}{\pi}$
(B) 0
(C) $\frac{16}{\pi}$
(D) $\frac{\pi}{16}$
(E) Some other value

$u = \frac{\pi x}{16}$; $du = \frac{\pi}{16} dx$

$x = 0..8$
$u = 0..\frac{\pi}{2}$

$$\int_0^8 \sin\left(\frac{\pi x}{16}\right) dx$$
$$= \frac{16}{\pi}\int_0^{\pi/2} \sin u \, du$$
$$= \frac{16}{\pi}\Big[-\cos u\Big]_0^{\pi/2}$$
$$= \frac{16}{\pi}\big(0 - (-1)\big)$$
$$= \boxed{\frac{16}{\pi}}$$

13. [5 pts] The graph of $f(x) = \dfrac{e^{\sqrt{x}}}{\sqrt{x}}$ is given below. What is the area of the region bounded by the graph of $f(x)$, the x-axis, and the lines $x = 1$ and $x = 4$?



(A) $e^2 - e$

(B) $2(e-1)$

(C) $e^4 - e$

(D) $2\left(e^2 - e\right)$

(E) $2\left(e^4 - e\right)$

$$A = \int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$$

$u = \sqrt{x}$ $\quad x = 1..4$

$du = \dfrac{1}{2\sqrt{x}}\,dx$ $\quad u = 1..2$

$$= 2\int_1^2 e^u\,du$$

$$= 2e^u\Big|_1^2$$

$$= \boxed{2(e^2 - e)}$$

14. [5 pts] If $F(x)$ is an antiderivative of $\sqrt{x} + 6x^2 + 6x^3$ and $F(1) = 0$, what is $F(0)$?

(A) $\dfrac{23}{6}$

(B) $-\dfrac{25}{6}$

(C) $-\dfrac{31}{6}$

(D) $\dfrac{25}{6}$

(E) The answer is not unique.

$$F(x) = \int\left(\sqrt{x} + 6x^2 + 6x^3\right)dx$$

$$= \frac{2}{3}x^{3/2} + \frac{\cancel{6}^{2}x^3}{\cancel{3}} + \frac{\cancel{6}^{3}x^4}{\cancel{4}_{2}} + C$$

$$0 = F(1) = \frac{2}{3} + 2 + \frac{3}{2} + C \Rightarrow C = -\frac{4+12+9}{6} = -\frac{25}{6}$$

$$\boxed{F(0) = C = -\frac{25}{6}}$$

15. [5 pts] If

$$\int_1^7 f(x)\,dx = 3 \quad \text{and} \quad \int_5^7 f(x)\,dx = 4,$$

find

$$\int_1^5 (1 + 2f(x))\,dx.$$

- (A) 2
- (B) 3
- (C) 4
- (D) 6
- (E) 7

$$\int_1^5 (1 + 2f(x))\,dx = \int_1^5 1\,dx + 2\int_1^5 f(x)\,dx$$

$$= \int_1^5 1\,dx + 2\Big[\underbrace{\int_1^7 f(x)\,dx}_{=3} + \underbrace{\int_7^5 f(x)\,dx}_{=-4}\Big]$$

$$= (5-1) + 2(3-4)$$

$$= \boxed{2}$$

16. [5 pts] Find

$$\frac{d}{dx}\left(\int_2^{\cos(x)} \frac{1-t^2}{1+t^2}\,dt\right).$$

- (A) $-\sin(x)$
- (B) $-\dfrac{\sin^2(x)}{1+\cos^2(x)}$
- (C) $-\dfrac{\sin^3(x)}{1+\cos^2(x)}$
- (D) $\dfrac{\sin^3(x)}{1+\cos^2(x)}$
- (E) $\dfrac{\sin^2(x)}{1+\cos^2(x)}$

FTC, part 1

$$\frac{d}{dx}\left(\int_0^{\cos(x)} \frac{1-t^2}{1+t^2}\,dt\right) = \frac{1-\cos^2 x}{1+\cos^2 x}\cdot(\cos x)'$$

$$= \frac{\sin^2 x}{1+\cos^2 x}(-\sin x)$$

$$= \boxed{-\frac{\sin^3 x}{1+\cos^2 x}}$$

Part II: Free Response

Unless specified otherwise, show all work leading to your final answer clearly and in a structured manner.

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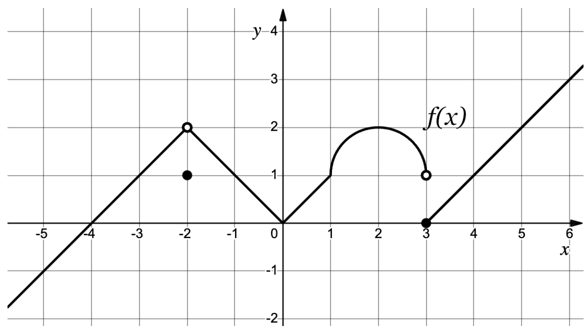
17. [20 pts] For this problem, the graph of $f(x)$ is given below. Note that all curves shown are either straight lines or arcs of a circle, and the ends of the graph extends linearly towards positive or negative infinity. **No need to show your work for this question.**



(2 PTS) (a) $\lim_{x \to -2} f(x)$

ANSWER

2

(2 PTS) (c) $\lim_{x \to 2} (5f(x) - 4x + e)$

$= 5 \cdot 2 - 4 \cdot 2 + e = 2 + e$

ANSWER

$2 + e$

(2 PTS) (b) $\lim_{x \to 3^-} f(x)$

ANSWER

1

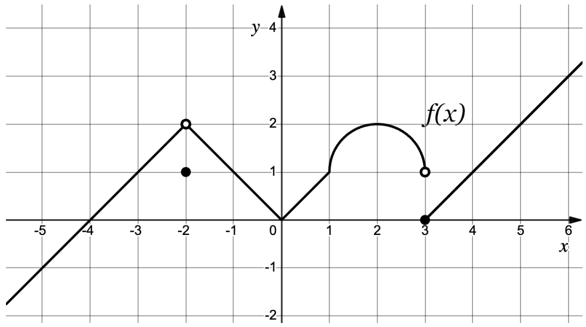
(2 PTS) (d) $\lim_{x \to -4} \dfrac{f(x) - f(-4)}{x - (-4)}$ $= f'(-4)$

$= 1$

ANSWER

1

The graph of $f(x)$ from the previous page is shown again:



(2 PTS) (e) List the x-coordinates of all critical points of $f(x)$:

ANSWER

$x = -2, 0, 1, 2, 3$

(2 PTS) (f) Find the average rate of change of $f(x)$ on the interval $[-2, 2]$.

$$a.r.o.c = \frac{f(2) - f(-2)}{2 - (-2)} = \frac{2 - 1}{4} = \frac{1}{4}$$

ANSWER

$\frac{1}{4}$

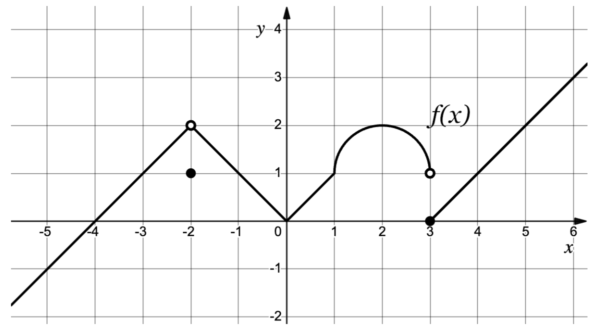
(2 PTS) (g) Find the instantaneous rate of change of $f(x)$ at $x = 2$.

At $x=2$, the tangent line is horizontal so i.r.o.c is 0.

ANSWER

0

The graph of $f(x)$ from the previous page is given again.



(2 PTS) (h) Can the Mean Value Theorem be applied on the interval $[-1, 1]$? **Justify.**

No, $f(x)$ is not differentiable at $x = 0$.

(2 PTS) (i) Can the Intermediate Value Theorem be applied on the interval $[0, 2]$? **Justify.**

yes, the function is continuous on $[0, 2]$ so IVT holds.

(2 PTS) (j) $\displaystyle\int_{-1}^{2} \underbrace{f(x)}_{} \, dx$ positive

$$= 2\,A(\triangle) + A(\square) + A(\text{quarter circle})$$

$$= 2 \cdot \frac{1 \cdot 1}{2} + 1 \cdot 1 + \frac{1}{4}\pi \cdot 1^2$$

$$= 2 + \frac{\pi}{4}$$

ANSWER

$2 + \frac{\pi}{4}$

18. [10 pts] Let $f(x) = \sqrt{x}$.

(4 PTS) (a) Using the limit definition of the derivative, and **no other method**, compute the derivative of $f(x)$.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\overset{[\frac{0}{0}]}{=} \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})}$$

$$= \boxed{\frac{1}{2\sqrt{x}}}$$

ANSWER

$$f'(x) = \frac{1}{2\sqrt{x}}$$

(2 PTS) (b) Check your answer in part (a) using the power rule.

$$f(x) = \sqrt{x} = x^{1/2}$$

$$\boxed{f'(x) = \frac{1}{2} x^{1/2 - 1} = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}}$$

(4 PTS) (c) Use linearization to estimate $\sqrt{25.04}$.

$$f(x) \approx L(x) = f(a) + f'(a)(x-a) \quad \text{where} \quad f(x) = \sqrt{x}; \; x = 25.04; \; a = 25$$

$$f(25) = \sqrt{25} = 5, \quad f'(25) = \frac{1}{2\sqrt{25}} = \frac{1}{10}$$

$$= 5 + \frac{1}{10}(25.04 - 25)$$

$$= 5 + \frac{1}{10} \cdot 0.04$$

$$= 5 + 0.004$$

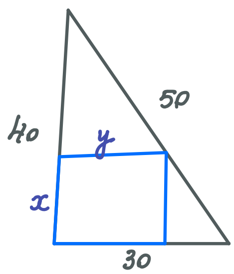
$$= 5.004$$

ANSWER

$$\sqrt{25.04} \approx 5.004$$

19. [10 pts] Consider a right triangle with side lengths 30, 40 and 50. A rectangle is inscribed in the triangle so that two of the sides of the rectangle lie along the legs (the sides forming the right angle) of the triangle. Determine the maximum possible area of the rectangle, and justify why this value represents an absolute maximum.

(2 PTS) (a) Make a sketch of the situation.



(4 PTS) (b) Write a formula for the area of the rectangle in terms of one variable along with a reasonable domain.

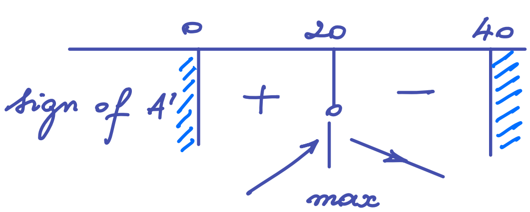
$A = xy$ where $\frac{y}{30} = \frac{40-x}{40} \Leftrightarrow 40y = 30(40-x) \Leftrightarrow y = \frac{3}{4}(40-x)$

$= \frac{3}{4}x(40-x)$

$= 30x - \frac{3}{4}x^2$ where $x \in (0, 40)$ or $[0, 40]$.

(4 PTS) (c) Solve the problem using calculus and justify your finding.

$A' = 30 - \frac{3}{2}x = 0 \Rightarrow x = 20$



The maximum possible area occurs when $x = 20$ and $y = \frac{3}{4}(40-20) = 15$ yielding $A = 20 \cdot 15 = 300$

ANSWER

$A = 300$

20. [20 pts] Consider the function

$$f(x) = \frac{-3e^x + 1}{e^x - 1},$$

with first derivative

$$f'(x) = \frac{2e^x}{(e^x - 1)^2},$$

and second derivative:

$$f''(x) = -\frac{2e^x(e^x + 1)}{(e^x - 1)^3}.$$

The function has a single x-intercept at $(-\ln 3, 0)$.

(2 PTS) (a) Find the domain of $f(x)$.

We want $e^x - 1 \neq 0$

$\Leftrightarrow e^x \neq 1$

$\Leftrightarrow x \neq 0$

So $\boxed{\text{domain of } f(x) = (-\infty, 0) \cup (0, \infty).}$

ANSWER

$(-\infty, 0) \cup (0, \infty)$

(3 PTS) (b) Find all vertical asymptotes of $f(x)$ or show that there are none. Justify your findings.

$x = 0$ is a vertical asymptote for $f(x)$ as:

$$\lim_{x \to 0^+} \frac{\overbrace{-3e^x + 1}^{-}}{\underbrace{e^x - 1}_{+}} \overset{\left[\frac{-2}{0}\right]}{=} -\infty$$

ANSWER

$x = 0$

(3 PTS) (c) The graph of $f(x)$ has a horizontal asymptote of $y = -3$ as $x \to +\infty$; find any additional horizontal asymptotes, or show that there are no others. Justify your answer.

$$\lim_{x \to -\infty} \frac{-\overbrace{3e^x}^{\to 0} + 1}{\underbrace{e^x}_{\to 0} - 1} = -1 \quad \therefore \quad y = -1 \text{ is another horizontal asymptote for } f(x).$$

ANSWER

$y = -1$

Recall that

$$f(x) = \frac{-3e^x + 1}{e^x - 1},$$

with first derivative

$$f'(x) = \frac{2e^x}{(e^x - 1)^2},$$

and second derivative:

$$f''(x) = -\frac{2e^x(e^x + 1)}{(e^x - 1)^3}.$$

(2 PTS) (d) Using the first and/or the second derivative of $f(x)$, find all critical points or show that no critical points exist.

$$f'(x) = \frac{2e^x}{(e^x-1)^2}$$

$= 0$ never

DNE $x = 0$: not in the domain

ANSWER

none

(2 PTS) (e) Using the first and/or the second derivative of $f(x)$, find all inflection points or show that no inflection points exist.

$$f''(x) = \frac{-2\overbrace{e^x}^{>0}\overbrace{(e^x+1)}^{>1}}{(e^x-1)^3}$$

$= 0$ never

DNE $x = 0$: not in the domain

ANSWER

none

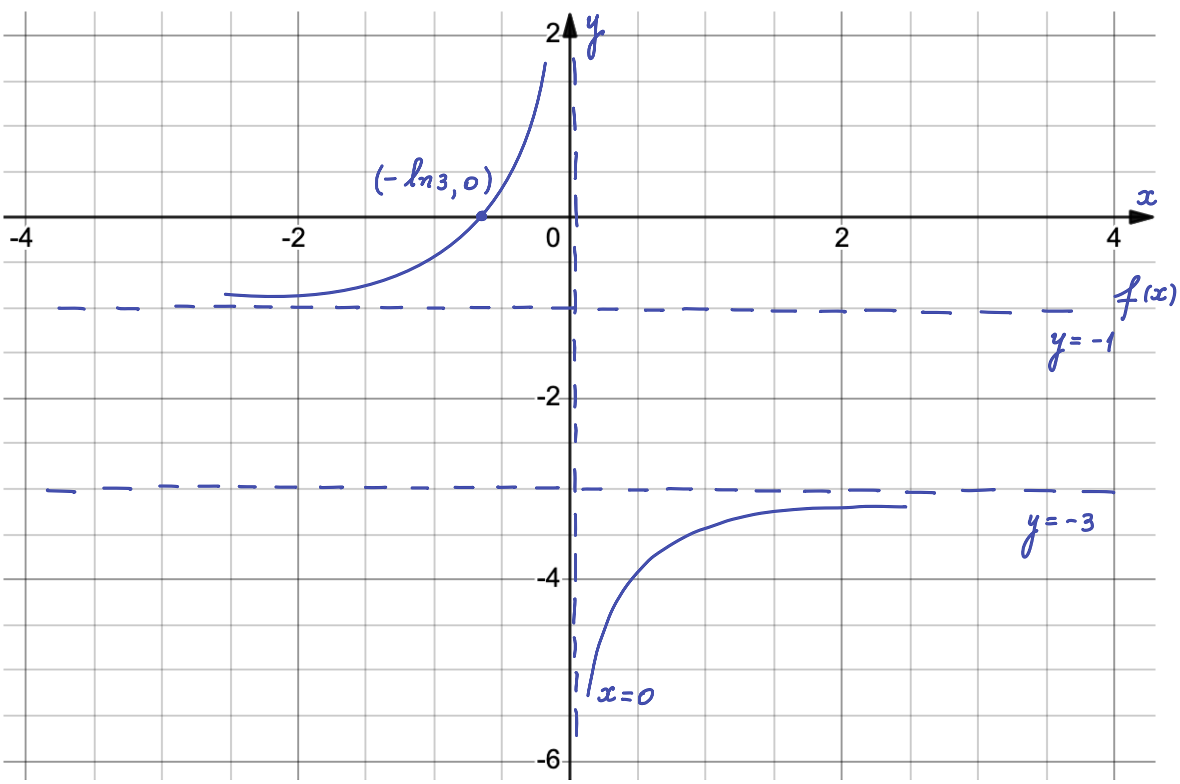
(4 PTS) (f) Organize your work in the table below. For each interval, indicate:

i. the sign of $f'(x)$ by writing + or −,
ii. whether the function is increasing or decreasing by drawing ↗ or ↘,
iii. the sign of $f''(x)$ by writing + or −,
iv. whether the graph is concave up or down by drawing ◡ or ⌒.

Use as many interval columns as needed.

Interval :	$(-\infty, 0)$	$(0, \infty)$	
Sign of f'	+	+	
Increasing/Decreasing	↗	↗	
Sign of f''	+	−	
Concave up/down	◡	⌒	

(4 PTS) (g) Sketch the function. Include intercepts, local extrema, inflection points and asymptotes, if any.



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