ALGEBRA QUALIFYING EXAM, FALL 2015

Instructions: Complete all 8 problems. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts.

(1) Let G be a group containing a subgroup H not equal to G of finite index. Prove that G has a normal subgroup which is contained in every conjugate of H, and which is of finite index.

(2) Let G be a finite group, H a p-subgroup and P a Sylow p-subgroup (p a prime). Let H act on the left cosets of P in G by left translation. Prove that there is an orbit under this action of length 1. Prove that xP is an orbit of length 1 if and only if H is contained in xPx^{-1}.

(3) Let R be a rng (ring without 1) which contains an element u such that for all $y \in R$, there is an $x \in R$ such that $xu = y$. Prove that R contains a maximal left ideal. Hint: imitate the proof (using Zorn's lemma) in the case when R does have 1.

(4) Let R be a PID (Principal Ideal Domain). Let $(a_1) < (a_2) < \cdots$ be an ascending chain of ideals in R. Prove that for some n, we have $(a_j) = (a_n)$ for all $j \geq n$.

(5) Let $u = \sqrt{2+\sqrt{2}}$, $v = \sqrt{2-\sqrt{2}}$, and $E = \mathbb{Q}(u)$.
  (a) Find (with justification) the minimal polynomial $f(x)$ of u over $\mathbb{Q}$.
  (b) Show $v \in E$. Show that E is a splitting field of $f(x)$ over $\mathbb{Q}$.
  (c) Determine the Galois group of E over $\mathbb{Q}$, and determine all the intermediate fields F with $E \supset F \supset \mathbb{Q}$.

(6) (a) Let G be a finite group. Show that there exists a field extension K/F with $\mathrm{Gal}(K/F) = G$. (You may assume that for any natural number n there is a field extension with Galois group S_n.)
  (b) Let K be a Galois extension of F with $|\mathrm{Gal}(K/F)| = 12$. Prove that there exists an intermediate field E of K/F with $[E : F] = 3$.
  (c) With K/F as in (b), does an intermediate field L necessarily exist satisfying $[L : F] = 2$? Give a proof or counterexample.

(7) (a) Show that two 3×3 matrices over the complex numbers are similar if and only if their characteristic polynomials are equal and their minimal polynomials are equal.
  (b) Does the conclusion of the previous part hold for 4×4 matrices? Justify your answer with a proof or counterexample.

(8) Let V be a vector space over a field F and V^* its dual. A symmetric bilinear form $(\cdot,\cdot)$ on V is a map $V \times V \to F$ satisfying the rules

$$(av_1 + bv_2, w) = a(v_1, w) + b(v_2, w) \text{ and } (v, w) = (w, v)$$

for $a, b \in F$ and $v_1, v_2, v, w \in V$. The form is nondegenerate if the only element w satisfying $(v, w) = 0$ for all $v \in V$ is $w = 0$.

Suppose $(\cdot,\cdot)$ is a symmetric nondegenerate bilinear form on V. If W is a subspace of V, define

$$W^\perp = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}.$$

(a) Show that if X and Y are subspaces of V with $Y \subset X$, then $X^\perp \subseteq Y^\perp$.

(b) Define an injective linear map $\psi : Y^\perp/X^\perp \hookrightarrow (X/Y)^*$ which is an isomorphism if V is finite-dimensional.