ALGEBRA QUALIFYING EXAM, FALL 2025

Instructions: Complete all 7 problems. Each problem is worth 10 points. Provide ample justification to your answers. In multi-part problems, you may assume the result of any part (even if you have not been able to do it) in working on subsequent parts.

(1) Let G, H be finite cyclic groups. Prove that $G \times H$ is cyclic if and only if $(|G|, |H|) = 1$.

(2) (a) Using Sylow theory show that every group of order 10 is not simple.
  (b) Classify all groups of order 10 and justify your answer. For the non-abelian group(s), give a presentation by generators and relations.

(3) Let A be an $n \times n$ matrix over a field k.
  (a) Define precisely what if means for A to be diagonalizable.
  (b) Provide the definition of an eigenvalue for A, the definition of an eigenspace, and a criterion via eigenspaces for A to be diagonalizable.
  (c) Prove that if A and B are two $n \times n$ matrices that are diagonalizable with the property that $AB = BA$ then they are simultaneously diagonalizable.

(4) Let $u \in \mathbb{C}$ be a root of the polynomial $m(x) = x^3 - 6x^2 + 9x + 3 \in \mathbb{Q}[x]$, and consider the field extension $\mathbb{Q}(u) \subseteq \mathbb{C}$ of $\mathbb{Q}$.
  (a) Show that $m(x)$ is irreducible in $\mathbb{Q}[x]$.
  (b) Define a ring homomorphism $f : \mathbb{Q}[x] \to \mathbb{C}$ with image $\mathbb{Q}(u)$. Conclude that there is an isomorphism between $\mathbb{Q}(u)$ and a quotient of $\mathbb{Q}[x]$, and specifiy that quotient ring explicitly.
  (c) What is the degree $[\mathbb{Q}(u) : \mathbb{Q}]$? Give a $\mathbb{Q}$-basis of $\mathbb{Q}(u)$.
  (d) Express the elements u^4 and u^{-1} in $\mathbb{Q}(u)$ in terms of the basis you gave in part (c).

(5) Let $F = \mathbb{Q}(\sqrt[3]{5})$
  (a) Find the smallest field K such that $F \subseteq K$ such that K is a Galois extension over $\mathbb{Q}$.
  (b) Prove that K is Galois over F.
  (c) Compute that Galois group $\mathrm{Gal}(K/\mathbb{Q})$ and $\mathrm{Gal}(K/F)$.

(6) Let R be a ring with unity.
  (a) Define what it means for an R-module to be torsion free.
  (b) Prove that if F is a free module, then any short exact sequence of R-modules $0 \to N \to M \to F \to 0$ splits.
  (c) Let R be a PID. Show that any finitely generated R-module M can be expressed as a direct sum of a torsion module and a free module. [You may assume that a finitely generated torsion free module over a PID is free.]

(7) Consider the matrix in $M_3(\mathbb{C})$

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 3 & 8 & -14 \\ 3 & 6 & -10 \end{bmatrix}$$

(a) Find the Jordan canonical form J of A.

(b) Find an invertible matrix P such that $P^{-1}AP = J$. (You should not need to compute P^{-1}.)