1. SOME ALGEBRA

(1) Assuming \( h \neq 0 \), what is \( \frac{f(x+h)-f(x)}{h} \) where \( f(x) = (x+1)^2 \)? Simplify.

(2) Find the domain of the function
\[
f(x) = \frac{\sqrt{x+2} + \log_2(5-x)}{x}.
\]

(3) (*) Consider the function \( f(x) = \ln \left( x + \sqrt{1 + x^2} \right) \). Find the domain of \( f \). Determine the parity of this function, i.e. is it odd, even, or neither?

(4) What is the equation of the secant line joining the points of the graph \( f(x) = 2^x \) whose \( x \)-coordinates are respectively 1 and 2?

(5) Find the point(s) of intersection of the hyperbolas \( x^2 + 3xy = 54 \) and \( xy + 4y^2 = 115 \).

2. LIMITS

Finding the limit at a real value without using l’Hôpital’s rule

\[
(6) \lim_{x \to 3} x^2 - 7x + 12 + \sqrt{x^2 - 5} = \quad (9) \lim_{x \to 0} \frac{x^4 + 5x - 3}{2 - \sqrt{x^2 + 4}} = \\
(7) \lim_{x \to 2} \frac{x^2 - 4x + 4}{x^2 - 5x + 6} = \quad (10) \lim_{x \to 1} \frac{x^3 - 1}{(x - 1)^2} = \\
(8) \lim_{x \to 4} \frac{3 - \sqrt{x + 5}}{x - 4} = \quad (11) (*) \lim_{x \to 0} x^4 \cos(2/x)
\]
Limits of trigonometric type

\[(12) \lim_{x \to 0} \frac{\sin^2 5x}{2x \tan 3x} = \quad (13) \lim_{x \to 0} \frac{\cos 2x - 1}{\cos x - 1} = \]

Limits at infinity

\[(14) \lim_{x \to -\infty} \frac{7}{x^3 - 4} = \quad (18) \lim_{x \to -\infty} \frac{7x^2 - x + 11}{4 - x} = \]
\[(15) \lim_{x \to \infty} \frac{10}{x^2 + 10} = \quad (19) \lim_{x \to \infty} \frac{\sqrt{x^2 - 5x}}{x + 3} = \]
\[(16) \lim_{x \to \infty} \frac{7x^2 + x - 100}{2x^2 - 5x} = \quad (20) (*) \lim_{x \to \infty} \left(\frac{x - 2}{x - 1}\right)^x = \]
\[(17) \lim_{x \to \infty} x - \sqrt{x^2 + 7} = \]

One sided limits

\[(21) \lim_{x \to 3^+} \frac{x^2 + 3x}{9 - x^2} = \quad (22) \lim_{x \to 3^-} \frac{x^2 - 3x}{x^2 - 9} = \]

3. Asymptotes

(23) The line \( y = mx + p \), with \( m \neq 0 \) is an oblique asymptote (or slant asymptote) of \( f(x) \) iff \( \lim_{x \to \infty} \frac{f(x)}{x} = m \) and \( \lim_{x \to \infty} f(x) - mx = p \). Show that \( f(x) = \sqrt{x^2 - 4x} \) has an oblique asymptote at \( \infty \) and a different one at \(-\infty\).

(24) (*) Show that if \( f(x) \) is a rational function then \( f(x) \) has an oblique asymptote iff the degree of the numerator is exactly one more than the degree of the denominator. [hint: how can you write \( f(x) \) after performing polynomial division?] Find the oblique asymptote(s) of \( f(x) = \frac{x^2 - 6x + 1}{x - 2} \) using (a) the above definition and (b) using long division.

(25) (*) Can a rational function have two distinct oblique asymptotes?
Find all asymptotes (vertical, horizontal and/or oblique) of the following functions

\[ e(x) = \frac{x^2 - 4x}{2x + 1} \]
\[ f(x) = \frac{2x + 1}{3x + 2} \]
\[ h(x) = \frac{x^4 + 1}{x^2 - 1} \]
\[ i(x) = \frac{x^3}{x^2 + 1} \]
\[ j(x) = 2x - \sqrt{4x^2 + 4} \]

(26) Find all the asymptotes (if any) to the function 

\[ f(x) = \frac{x^2 - 1}{x|x + 1|} \]

(27) (*) Consider the function 

\[ f(x) = ax - \sqrt{bx^2 - 1} \] where \( a \) and \( b \) does this function have an oblique asymptote of slope 5 at \(-\infty\) and of slope 1 at \(+\infty\)?

4. Derivatives

(33) Using the limit definition, compute \( f'(3) \) where \( f(x) = x^2 + \frac{2}{x} \)

Compute the derivatives of the following functions:

\[ f(x) = 4x^5 - 5x^4 \]
\[ g(x) = 3x^2(x^3 + 1)^7 \]
\[ h(x) = (3x - 1)^2 \]
\[ i(x) = \frac{(3x^2 + x)^2}{x^2 + 2x} \]
\[ j(x) = (\arctan(2x))^{10} \]
\[ k(x) = x^7(x^2 - x)^5\sin^4(x^2)e^{4x} \]
\[ l(x) = \arcsin(2^\sin x) \]
\[ m(x) = \log_5(3x^2 + x) \]
\[ n(x) = \frac{3\sin(x) + 2}{4\sin(x) + 3} \]

(34) \[ f(x) = 4x^5 - 5x^4 \]
(35) \[ g(x) = 3x^2(x^3 + 1)^7 \]
(36) \[ h(x) = (3x - 1)^2 \]
(37) \[ i(x) = \frac{(3x^2 + x)^2}{x^2 + 2x} \]
(38) \[ j(x) = (\arctan(2x))^{10} \]
(39) \[ k(x) = x^7(x^2 - x)^5\sin^4(x^2)e^{4x} \]
(40) \[ l(x) = \arcsin(2^\sin x) \]
(41) \[ m(x) = \log_5(3x^2 + x) \]
(42) \[ n(x) = \frac{3\sin(x) + 2}{4\sin(x) + 3} \]

(43) Determine the following limit quickly: \( \lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x - 2} \).

(44) Find \( f'(3\pi) \) where \( f(x) = (\cos x + 1)^x \).

(45) If \( f(2) = 3 \), \( g(2) = 4 \), \( g(3) = 2 \), \( f'(2) = 5 \) and \( g'(3) = 2 \) find

\[ \left( \frac{f(g(x)) + x}{f^2(2x - 4)} \right)' \]

at \( x = 3 \).

(46) Find \( \frac{dy}{dx} \) where \( y \) is a differentiable function satisfying \( \frac{\sin y}{y^2 + 1} = 3x \).

5. Tangents

(47) Find the point of intersection of the lines tangent to the graph of \( f(x) = x\sin(x) \)
at \( x = \frac{\pi}{2} \) and \( x = \pi \).
99 PROBLEMS

48 Find the tangent(s) to the graph of \( f(x) = x^2 - 2x + 1 \) passing through the point \((4, 1)\).

49 Find the equation of the line tangent at \((1, 1)\) to the graph of the function
\[
y^3 + xy = x^3 - x + 2.
\]

50 (*) (Legendre Transform) Consider a smooth convex function \( f(x) \). Pick a slope \( m \) and let \( f^*(m) \) be the y-intercept of the tangent to the graph of \( f(x) \) whose slope is \( m \). Find the function \( f^*(m) \) where \( f(x) = x^2 - 2x \).

6. Extrema & Concavity

51 The function \( f(x) = a \ln x - a^3 x \) has a local minimum at \( x = 4 \) for \( a \neq 0 \). What is \( a \)?

52 Over which interval is \( f(x) = x^3 - 6x^2 + 3x \) (a) concave up? (b) decreasing?

7. Study of functions

Study the following functions. I.e. find the (1) domain, (2) asymptotes and/or discontinuities, study the (3) growth and (4) concavity; locate (5) all extrema and inflection points; (6) find the roots and (7) sketch the graph

53 \( x^3 - 3x^2 \)

54 \( x^4 - 2x^3 \)

55 \( 3x + 4 \)

56 \( 2x + 3 \)

57 \( x^3 \)

58 \( x^2 - 4 \)

59 \( x(x-3)^2 \)

60 \( x^2 - 1 \)

61 \( 3 \left( \sqrt{x^2 - 1} - x \right) \)

62 \( \frac{1}{x} - \frac{1}{x(1-x)} \)

63 \( \frac{x}{x - 2} \)

64 \( \frac{x - 3}{|x - 2|} \)

65 \( \frac{x}{3x^3 - 2x} \)

66 (*) Consider the function \( f(x) = \frac{1}{x^2 - 3x + 2} \). Study and sketch the function. Using
the previous graph, plot (a) \( \phi(x) = e^{f(x)} \) and, (b) \( \psi(x) = f(|x|) \).

8. Varia

67 Give a lower bound on the number of roots of \( f(x) = \cos(\pi x)/x \) on the interval \([1, 3]\). [hint: Intermediate value theorem]

68 Suppose that a function \( f(x) \) has a maximum at \( x = 3 \). True or False? Justify.

- The function \( f^2(x) \) has a maximum at 3.
- The function \( e^{f(x)} \) has a maximum at 3.
- The function \( f(x - 3) \) has a maximum at 0.
(69) Without a calculator estimate \( \sin^2 \left( \frac{99\pi}{4} \right) \).

(70) If \(-1 \leq f'(x) \leq 3\) for all \(x\) in \([1, 4]\) and \(f(2) = 4\), find the maximal and minimal possible values of \(f(4)\).

(71) (**) Suppose that \(f : [0, 1] \to [0, 1]\) is a continuous function. Prove that \(f\) has a fixed point in \([0, 1]\), i.e., there is at least one real number \(x\) in \([0, 1]\) such that \(f(x) = x\).

(72) (**) Suppose that \(g\) is a continuous function on \([0, 2]\) satisfying \(f(0) = f(2)\). Show that there is at least one real number \(x\) in \([1, 2]\) with \(f(x) = f(x - 1)\).

(73) Suppose that \(\sum_{i=1}^{10} a_i = 100\) compute \(\sum_{i=1}^{10} (2a_i + 3 - i)\).

9. RELATED RATES

(74) A 10 ft ladder is leaning against the wall. How fast is the bottom of the ladder sliding when the top part is 3 ft above the ground and gliding at a rate of 1 ft per second.

(75) A conical cup has a diameter of 4 cm and a height of 8 cm. How fast is the level dropping when the height is 4 cm and the water escapes from the bottom at a rate of 1 cm\(^3\) per second.

10. OPTIMIZATION

(76) Find the maximal area of rectangle whose sides are parallel to the coordinate axes and whose vertices lie on the curve of equation \(x^2 + y^4 = 1\).

(77) We have 12 m\(^2\) of material to make a box whose bottom is square and sides are rectangular (the box has no top). What is the maximal volume that such a box can have?

11. INTEGRATION

(78) Using 4 rectangles and the right endpoint method estimate \(\int_{0}^{12} \frac{2}{x^2 + 2} \, dx\).

(79) Compute the area under the graph of \(g(x) = x + 3x^3 - \sin(2x) + xe^{-x^2} + x^2\) over the interval \([-3, 3]\).

Compute the following integrals

\[
\begin{align*}
\int_{0}^{1} x e^{-x^2} \, dx & \quad \text{(80)} \\
\int (\sin x + \cos x)^2 \, dx & \quad \text{(81)} \\
\int_{0}^{1} \frac{x^4 - 3x^2}{x^2} \, dx & \quad \text{(82)} \\
\int_{-2}^{3} |x - 1| \, dx & \quad \text{(83)} \\
\int \frac{x^3}{x^2 + \pi} \, dx & \quad \text{(84)} \\
\int 5^{2x} \, dx & \quad \text{(85)}
\end{align*}
\]
(86) (*) \[ \int_0^1 \frac{x}{\sqrt{x+1}} \, dx \]
(87) (*) \[ \int \frac{1}{1+e^x} \, dx \]
(88) (** \[ \int \frac{1}{1+\sin^2(x)} \, dx \]

(89) Compute the area of the region between the parabolas \( y = 2x^2 - 2 \) and \( y = x^2 + x \).
(90) Compute the area of the region bound by \( y = x^3 + x, \ y = x^3, \ x = -2 \) and \( x = 1 \).

12. **Fundamental Theorem of Calculus**

(91) Find \( f'(x) \) where \( f(x) = \int_x^{x^2} \frac{\sin t}{t} \, dt \).

13. **Graph analysis**

Based on the above picture representing the graph of \( f(x) \), answer the following questions.

(92) \( \lim_{{x \to 1^+}} f(x) = \)
(93) \( \lim_{{x \to 3}} f(x) = \)
(94) (*) \( \lim_{{x \to -2^+}} f(-x) = \)
(95) \( f'(\frac{3}{2}) = \)

(96) \[ \int_0^3 f(x) \, dx \]
(97) \( F'(4) = \) where \( F(x) = \int_0^x f(t) \, dt \)
(98) Sketch \( f'(x) \)
(99) Sketch \( F(x) = \int_1^x f(t) \, dt \)

\( \diamond \diamond \diamond \)