QUALIFYING EXAM IN COMPLEX ANALYSIS, SPRING 2017

Test-taking tips:

1. It is to your advantage to begin work each problem, even if you don't see how to finish it.

2. Problems will be equally weighted.

3.The more elementary your arguments, the better.

1. Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(x + iy) = x^2 - y^2 + 3y + i(xy - 4y)$. Find all points in $\mathbb{C}$ at which f is complex differentiable.

2. Show that there is a holomorphic function $w = f(z)$ on the open set $|z| > 2$, such that $w^2 = z^2 - 1$.

3. Exhibit a sequence of explicit conformal mappings, the composite of which gives a one-to-one conformal map from the region

$$R = \{z \ : \ |z| < 2 \ \text{ and } |z - 1| > 1\}$$

onto the upper half plane.

4. Let $U \subset \mathbb{C}$ be the punctured unit disk, $0 < |z| < 1$, and let V be the open annulus $1/2 < |z| < 1$.

i) Prove that there is a smooth bijection $f : U \to V$ with a smooth inverse. (One way to do this is to write one down.)

ii) Prove however that no such function f is conformal.

5. Compute by residues: $\displaystyle\int_0^\infty \frac{\sin(x)}{x(x^2+1)}dx$

6. Show that the equation $\sin(z) = e^\alpha z^3$, for $\alpha > 1$ has exactly three solutions inside the unit disk.

7. Describe all entire functions $f(z)$ such that

$$|f(z)| \leq |\sin(z)| \quad \text{for all } z.$$