PHD PRELIMINARY EXAM IN COMPLEX ANALYSIS
FALL 1991

Work any 6 problems. Do all parts unless otherwise indicated.

1. a) Show that if $f:\mathbb{C}\to\mathbb{C}$ is holomorphic at a, i.e. $\lim \{ f(z)-f(a) \} / (z-a)$ exists, as $z\to a$, then f satisfies the Cauchy-Riemann equations at a.

   b) Assume $f:\mathbb{C}\to\mathbb{C}$ is differentiable at a in the "real sense", i.e. there is an $\mathbb{R}$-linear map $T:\mathbb{C}\to\mathbb{C}$ such that $\lim \{ f(z)-f(a) - T(z-a) \} / |z-a| = 0$, as $z\to a$. Prove that if f satisfies the Cauchy - Riemann equations at a, then the map T is also $\mathbb{C}$-linear.

2. a) Find the image of the region $\{ 0 \le \mathrm{Im}(z) \le 2 \}$ under the map $w = (z+i) / (z-i)$, and show the images of the lines $y = b$ for $0 \le b \le 2$.

   b) Find the image of the region $\{ 0 \le \mathrm{Re}(z) \le \pi/2 \}$ under the map $w = \sin(z)$, and indicate the image of the lines $x = a$ for $0 \le a \le \pi/2$. [You might start by finding real and imaginary parts of sin(z).]

3. a) Assume $w = f(z) = \Sigma\, a_n(z-1)^n$, $n = 0,\ldots,\infty$, is a power series with radius of convergence $\rho = 1$, and which can be continued analytically along every path γ beginning at $z = 1$ and lying in $U = \mathbb{C} - \{0\}$. Ignoring points at infinity, tell how to construct (in theory) the Riemann surface X associated to f, as a possibly branched cover of an open set V in $\mathbb{C}$, and describe all the different possible such X's and V's that may result.

OR b) Let $w = f(z) = 1 + \Sigma\, a_n z^n$, $n = 1,\ldots,\infty$, be the unique power series with $w^2 = 1 - z^3$, and $f(0) = 1$. F determines a compact Riemann surface X which is a finite-sheeted branched cover of the Riemann sphere S. Determine (i): the number of sheets, (ii) the branch points on S, (iii) the topological structure of X, and (iv) the result of analytically continuing f around the curve γ beginning at the origin, going up the y axis to the point 2i, then once counterclockwise around the circle of radius 2, then back down to the origin.

4. a) If $f:U\to\mathbb{C}$ is an analytic function in an open, connected set U in $\mathbb{C}$, and if the zeroes of f have an accumulation point p in U, then prove f is constant in U.

   b) Give an example of a non - constant function holomorphic in the open unit disc U, and having an infinite number of zeroes there, or prove this is impossible.

5. a) Assuming Cauchy's integral formula for the values of a holomorphic function, deduce that an entire holomorphic function is entire analytic (i.e. has an everywhere convergent power series expansion), and give Cauchy's formulas for the coefficients of the expansion.

   b) Deduce Liouville's theorem that an entire, bounded, holomorphic function is constant.

6. Let ω_1, ω_2, be complex numbers with non - real ratio. Prove the existence of a non constant meromorphic function on $\mathbb{C}$ with periods $\omega = n_1\, \omega_1 + n_2\, \omega_2$, for all integers n_1, n_2.

7. a) Let U be a simply-connected subset of $\mathbb{C}$, and let $f{:}U \to U$ be a holomorphic conformal isomorphism. Prove that if f has two fixed points, then f is the identity, or give a counterexample.

b) Prove any holomorphic injection $f{:}\mathbb{C} \to \mathbb{C}$ is also a surjection or give a counterexample.

8. a) Prove that a C^2 harmonic function $u{:}\ \mathbb{C} \to \mathbb{R}$ is the real part of an entire holomorphic function $f{:}U \to \mathbb{C}$.

b) Deduce the maximum modulus principle for harmonic functions from an appropriate theorem about holomorphic functions.