Complex A

Complex Analysis Prelim

Fall, 1993

1. Evaluate the following integrals using methods of complex analysis:

a) $\displaystyle\int_0^\infty \ln(x)/(1+x^2)\,dx$ b) $\displaystyle\int_0^{2\pi} 1/(3+2\cos(\theta))\,d\theta$

2. a) Compute the Taylor series expansion of $f(z) = e^z/(1+z^2)$ about 0 converging in the disc $|z| < 1$, and its Laurent series expansion about 0 converging in the annulus $|z| > 1$.

b) Let Γ be the rectangle with corners at $(0, \pm 2)$, $(2, \pm 2)$, traversed counterclockwise, and let $w = f(z) = z^5 - 1$. Compute

$$(2\pi i)^{-1} \int_{f(\Gamma)} 1/w\, dw.$$

3. a) Show that the power series $\sum_{n=0}^{\infty} z^{2^n}$ converges for $|z| < 1$, and cannot be analytically continued beyond the circle $C(0,1)$.

b) Describe the Riemann surface of the multi-valued function

$$w = \sqrt{z(z^2-1)}.$$

4. a) Give Goursat's proof of the following version of Cauchy's theorem:

If $R \subseteq \mathbb{C}$ is a rectangle, if $f(z)$ is defined in a neighborhood of R, and if $f'(z)$ exists for all $z \in R$, then $\int_{\partial R} f(z)dz = 0$.

b) State the Fundamental Theorem of Algebra, and give any proof of it using complex analysis.

5. a) Let $D = \{z \in \mathbb{C} : -\pi/2 < \mathrm{Re}(z) < \pi/2,\ \mathrm{Im}(z) > 0\}$. Describe the image of D under the map $w = \sin(z)$.

b) Let $D = \{z \in \mathbb{C} : |z + i| < \sqrt{2},\ \mathrm{Im}(z) > 0\}$. Exhibit a conformal mapping taking D to the upper halfplane, whose extension to ∂D fixes the points $-1, 0, 1$.