Comp A

Complex Analysis Prelim — May 12, 1994

Closed Book, 3 Hours, Do EIGHT of the ten problems

1. Let $f: \mathbf{C} \to \mathbf{C}$ be analytic and satisfy

$$|f(z)| \le C(1+|z|)^N$$

for some constant C whenever $|z| \ge R_0$. Give the general form for f and prove that it is the general form.

2.

(a) Demonstrate the existence of some $\epsilon > 0$ so that

$$f(z) = \begin{cases} z/(e^z - 1), & z \ne 0 \\ 1, & z = 0 \end{cases}$$

is analytic in $N_\epsilon(0)$.

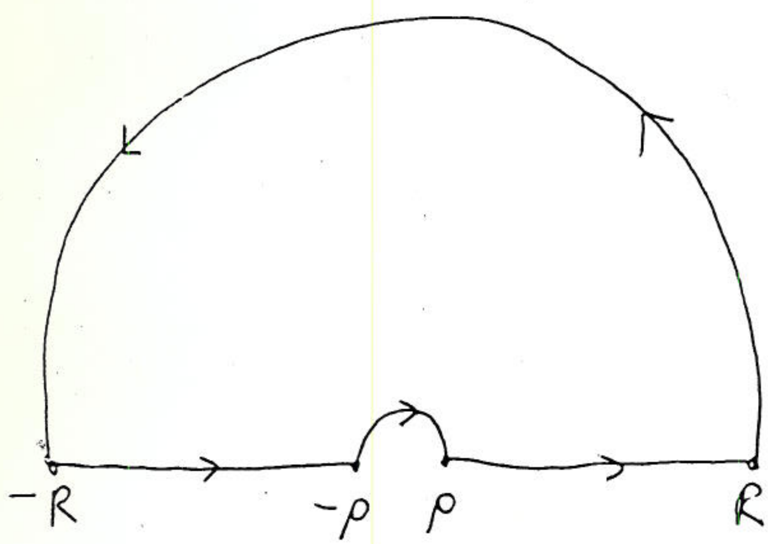
(b) Demonstrate rigorously that the radius of convergence of the power series for f about 0 is 2π.

3. Let

$$I(\alpha) = \int_0^\infty \frac{z^\alpha}{1+z^2}\, dz$$

(a) For which $\alpha \in \mathbf{C}$ does the integral make sense?

(b) Calculate $I(\alpha)$ using the path:



No change of variables is necessary.

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6. Construct an entire function such that $f(n) = n$ and $f'(n) = 0$, for each $n \in \mathbb{Z}$.

7. Do either A) or B):

   A) Let $f(z)$ be an entire function which is real-valued on the edges of the triangle with corners $0, 1, \frac{1}{2} + \frac{1}{2}\sqrt{3}i$. Show that $f(z)$ must be constant. Generalize.

   B) Find all entire functions such that $|f(z)| = 1$ whenever $|z| = 1$.

8. Explain why there exists a function $u(z)$ on $|z| \leq 1$, harmonic in $|z| < 1$, such that $u(e^{i\theta}) = |\sin(\theta)|$ for all θ. Evaluate $u(0)$ and $u(\frac{1}{2})$.

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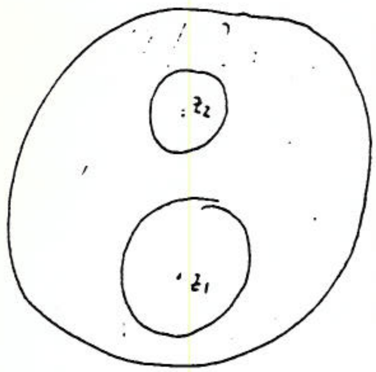
4. Let

$$U = \{z \in \mathbf{C};\ |z - z_0| < R, |z - z_1| > r_1, |z - z_2| > r_2\}$$

where $r_1 + r_2 < |z_1 - z_2|$, $R - r_1 > |z_1 - z_0|$, $R - r_2 > |z_2 - z_0|$.



Let $f\colon U \to \mathbf{C}$ be analytic. Imitating the development in an annulus, show that

$$f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \sum_{m=1}^{\infty} b_{-m}(z - z_1)^{-m} + \sum_{m=1}^{\infty} c_{-m}(z - z_2)^{-m}$$

5. State and prove the analogue of Schwarz' lemma that holds under the hypotheses that $f\colon N_1(0) \to N_1(0)$ is analytic and $f(0) = f'(0) = 0$.

6. Let $\alpha \in (0, 2\pi]$ and let

$$\begin{aligned}
U_1 &= \{re^{i\theta};\ 0 < r < 1, 0 < \theta < \alpha\} \\
U_2 &= \{z \in \mathbf{C};\ |z| < 1, \operatorname{Im} z > 0\} \\
U_3 &= \{z \in \mathbf{C};\ \operatorname{Re} z, \operatorname{Im} z > 0\} \\
U_4 &= \{z \in \mathbf{C};\ \operatorname{Im} z > 0\} \\
U_5 &= \{z \in \mathbf{C};\ |z| < 1\}\ .
\end{aligned}$$

Find analytic bijections $U_1 \to U_2$, $U_2 \to U_3$, $U_3 \to U_4$, and $U_4 \to U_5$. (Powers and linear fractional transformations suffice.)

7. Let $P(z) = z^N + a_{N-1}z^{N-1} + \ldots a_0$.

(a) Use the change of variables $w = 1/z$ to show that

$$\frac{1}{2\pi i}\int_{|z|=R} \frac{P'(z)}{P(z)}\,dz = N$$

for R sufficiently large.

(b) Use the argument principle to draw a conclusion from (a).

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8. Use the function

$$\frac{1}{z^2 \sin z}$$

to calculate $\sum_{n=1}^{\infty}(-1)^n/n^2$.

9. Let f be a nonconstant doubly periodic meromorphic function on $\mathbf{C}$. Show that f must have at least two poles (counting multiplicity) in a fundamental domain (fundamental parallelogram.)

10. Let Λ and Λ' be two lattices in $\mathbf{C}$. Let $f\colon \mathbf{C}/\Lambda \to \mathbf{C}/\Lambda'$ be an analytic map of Riemann surfaces. Show that the form of f must be

$$f(z+\Lambda) = \alpha z + \beta + \Lambda'$$

for some constants α and β. (Use covering maps and problem 1.)