Algebra Preliminary Exam

September 1992

Work as many problems as possible.

1. Let G be the alternating group A_5. Let $H \subset G$ be the normalizer of a Sylow 5-subgroup. Show that $H = D_5$, the dihedral group of order 10.

2. State and prove the spectral theorem for a normal operator on a finite dimensional complex vector space.

3. Let R be a commutative ring, and let $R[[x]]$ be the ring of formal power series with coefficients in R. Assume that R is Noetherian, and let $f \in R[[x]]$. Prove that if all the coefficients of f are nilpotent, then f is nilpotent.

4. State and prove the dimension formula for a tower of field extensions $F \subset K \subset E$.

5. Let $F \subset E$ be a Galois field extension, and let $\zeta \in E$. Regarding E as a vector space over F, let $E \stackrel{l_\zeta}{\to} E$ be the linear transformation "multiplication by ζ". Then let $f_\zeta(x) \in F[x]$ be the characteristic polynomial of l_ζ.
   (a) Show that there is an irreducible monic polynomial $p(x) \in F[x]$ and an integer n such that $f_\zeta(x) = p(x)^n$. (Hint: Consider first the case that $E = F(\zeta)$.)
   (b) Define two functions of ζ:
   $$D_{E/F}(\zeta) = \det(l_\zeta)$$
   $$N_{E/F}(\zeta) = \prod_{r \in Gal(E/F)}$$
   (The latter function is called the norm of ζ.) Show that for all $\zeta, D_{E/F}(\zeta) = N_{E/F}(\zeta)$. (Hint: Establish a formula for $p(x)$ of part a in terms of the Galois group.)

6. Give an example of a commutative ring R, together with R-modules, A,B,C, and D, and give an exact sequence
   $$0 \to A \stackrel{\alpha}{\to} B \stackrel{\beta}{\to} C \to 0,$$
   such that upon tensoring with D, the sequence ceases to be exact. Explain.