Preliminary Exam in Algebra

September 19, 1994

Do as many problems as you can; each problem is worth 10 points. All rings have a multiplicative identity 1.

1. State and prove the Fundamental Homomorphism Theorem for groups.
2. Prove that there is no simple group of order 72.
3. Prove that every artinian integral domain is a field. (This generalizes the well-known result that every finite integral domain is a field.)
4. (a) Define the Jacobson radical $J(R)$ of a ring R.

   (b) Let R be a P.I.D. Prove that $J(R) = (0)$ if and only if R has infinitely many prime ideals.
5. Give an example of each of the following. Include a brief (1- or 2-line) explanation.

   (a) a ring R and a quadratic polynomial $f(x) \in R[x]$ having more than two roots in R;

   (b) a ring R and an ideal $I \subset R$ such that R/I is an integral domain but R is not;

   (c) a U.F.D. which is not a P.I.D.;

   (d) an indecomposable module which is not irreducible.
6. Prove that, as a $\mathbb{Z}$-module, $\mathbb{Q}$ is flat but not projective. (Recall that a right R-module M is flat if $M \otimes_R -$ is an exact functor.)
7. Let $f(x) = x^4 - 3 \in \mathbb{Q}[x]$.

   (a) Find the splitting field K of $f(x)$ over $\mathbb{Q}$, and compute the degree $[K : \mathbb{Q}]$

   (b) Compute the Galois group G of f over $\mathbb{Q}$.

   (c) Find all subgroups of G, and match them to the corresponding intermediate fields between $\mathbb{Q}$ and K.
8. Let F be a field, and $f(x) \in F[x]$ an irreducible polynomial of prime degree p. Suppose that E is a finite extension of F with $f(x)$ reducible in $E[x]$. Prove that $p \mid [E : F]$.
9. Let T be a Hermitian operator on a finite dimensional complex inner product space, with the property that $T^k = I$ for some positive integer k. Prove that, in fact, $T^2 = I$.

10. Let $A = \begin{bmatrix} 2 & 0 & 0 \\ 4 & 6 & 1 \\ -16 & -16 & -2 \end{bmatrix}$

(a) Find the Jordan canonical form J of A.

(b) Find an invertible matrix P such that $P^{-1}AP = J$. (You should not need to compute P^{-1}.)

(c) Write down the minimal polynomial of A.