Ph.D. Prelim: Probability Theory, August 2000

(Solve at least 5 problems completely.)

1. (a) If X and Y are independent random variables and $F(y)$ is the distribution function of Y, show that
$$P(Y \leq X) = E[F(X)].$$
(b) Suppose that X and Y are independent random variables with the same exponential density $f(x) = \theta e^{-\theta x}$. Show that the sum $X+Y$ and the ratio X/Y are independent.

2. (a) Show that every real valued characteristic function $\phi(t)$ satisfies the inequality
$$1 + \phi(2t) \geq 2[\phi(t)]^2.$$
(b) Show, for any two random variables X and Y with $\mathrm{Var}(X) < \infty$, that
$$\mathrm{Var}(X) = E[\mathrm{Var}(X|Y)] + \mathrm{Var}[E(X|Y)].$$

3. Show that random variables X_n, $n \geq 1$, and X satisfy $X_n \to X$ in distribution iff
$$E[F(X_n)] \to E[F(X)]$$
for every continuous distribution function F.

4. (a) Quote without proof a WLLN.
(b) Prove the Weierstrass Approximation Theorem using a LLN.

5. If $\{X_n\}$ are iid random variables with $E|X_1|^p = \infty$ for some $p \in (0,2)$, show that
$$P\left(\limsup_n \frac{S_n}{n^{\frac{1}{p}}}\right) = 1.$$

6. (a) Let X_n and Y_n be sequences of random variables. Suppose $X_n \to X$ in distribution and $Y_n \to c$ in probability, where c is a nonzero constant. Show that $X_n/Y_n \to X/c$ in distribution.
(b) Let $\{X_n,\ n \geq 1\}$ be a sequence of iid random variables with $EX_n = 0$ and $\mathrm{Var}(X_n) = 1$, for all $n \geq 1$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Show that, as $n \to \infty$,
$$Y_n = \frac{S_n}{\sqrt{\sum_{i=1}^n X_i^2}} \to N(0,1), \text{ in distribution.}$$

7. (a) State (without proof) the Doob's maximum inequality and Kolmogorov's inequality.
(b) Let $\mathcal{F}_n$ be a family of σ-algebras such that
$$\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots$$
and X be an integrable random variable. Show that
$$E[X|\mathcal{F}_n] \to E[X|\mathcal{F}_\infty] \text{ a.s. and in } L^1,$$
where $\mathcal{F}_\infty = \cap_{n=1}^\infty \mathcal{F}_n$.