Preliminary Exam in Algebra

August 2001

Problem 1. Prove that there are no simple groups of order 56.

Problem 2. Let p be a prime, $1 \leq n \leq p-1$ an integer and G be a p-subgroup of S_{np}. Prove that G is commutative. If you cannot do the general case, do it for $n = 2$.

Problem 3.
(a) How many elements of the group S_n commute with the element $(12)(34)$?
(b) What is the order of the group $\mathrm{GL}(n, \mathbb{F}_p)$ for a prime p?

Problem 4. Find the Galois group of the extension $\mathbb{Q}[\sqrt{2} + \sqrt{3} + \sqrt{5}]/\mathbb{Q}$.

Problem 5. Factor the polynomial $x^{16} - x$ into irreducibles in the field $\mathbb{F}_4$. Prove that you factors are irreducible.

Problem 6. Let R be a ring, and I an ideal of R. Suppose that every element of R which is not in I is a unit of R. Prove that I is a maximal ideal and moreover that it is the only maximal ideal of R.

Problem 7. Let R be a commutative ring with identity. Assume that R contains no zero-divisors and that R satisfies the descending chain condition on ideals. Prove that R is a field.

Problem 8. Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}.$$

Problem 9. Let A and B be two $n \times n$ matrices with complex coefficients. Assume that $(A - I)^n = 0$ and that $A^k B = BA^k$ for some $k \in \mathbb{N}$. Prove that $AB = BA$. (Hint: prove that A is a polynomial function of A^k.) Give a counterexample to this conclusion if $\mathbb{C}$ is replaced by field of positive characteristic.