Qualifying Examination of Numerical Analysis, 2005

Name______________________________ Student Id. No.______________

Instruction: *The following are ten problems in total. Please start each problem on a separate sheet of paper, write on only one side of the paper, and number each page. The time limit on this exam is three hours.*

[1] Suppose that a square matrix A is strictly diagonally dominant. Show that when applying Gauss-Jacob iteration to solve $Ax = b$, the iteration converges.

[2] Let A be an invertible matrix and $\tilde{A}$ be a perturbation of A satisfying $\|A^{-1}\|\,\|A-\tilde{A}\| < 1$. Suppose that x and $\tilde{x}$ are the exact solutions of $Ax = b$ and $\tilde{A}\tilde{x} = \tilde{b}$, respectively. Show that

$$\frac{\|x-\tilde{x}\|}{\|x\|} \le \frac{\text{cond}(A)}{1-\ \text{cond}(A)\frac{\|A-\tilde{A}\|}{\|A\|}}\left[\frac{\|A-\tilde{A}\|}{\|A\|} + \frac{\|b-\tilde{b}\|}{\|b\|}\right].$$

[3] Let U, Σ, V be the singular value decomposition(SVD) of A. Let

$$A^+ = V\Sigma^+U^T$$

be the pseudo inverse of A, where $\Sigma^+ = \text{diag}(\frac{1}{\sigma_1}, \frac{1}{\sigma_2}, \cdots, \frac{1}{\sigma_r}, 0, \cdots, 0)$ and r stands for the rank of A. Show that

$$AA^+A = A \text{ and } (A^+A)^T = A^+A.$$

[4] Let x and y be two vectors in $\mathbf{R}^2$. Suppose that $\|x\|_2 = \|y\|_2$. Show that there exists a unitary matrix H such that $Hx = y$.

[5] Suppose that p is a root of multiplicity $m > 1$ of $f(x) = 0$. Show that the following modified Newton's method

$$p_{n+1} = p_n - \frac{mf(p_n)}{f'(p_n)}$$

gives quadratical convergence.

[6] Show that the fourth order finite difference of f approximates $f^{(4)}$ at the order $O(h^2)$; i.e.,

$$\frac{f(a_4) - 4f(a_3) + 6f(a_2) - 4f(a_1) + f(a_0)}{h^4} - f^{(4)}(a_2) = \frac{h^2}{6}f^{(6)}(\xi)$$

for some $\xi \in [a_0, a_4]$, where $a_i = a + ih, i = 0, \cdots, 4$.

[7] Suppose that $f \in C^4[a, b]$. Prove the following error estimate for the Simpson rule. ie.,

$$\int_a^b f(x)dx - \frac{(b-a)}{6}(f(a) + 4f(\frac{a+b}{2}) + f(b)) = -\frac{(b-a)^5}{2880}f^{(4)}(\xi)$$

for some $\xi \in [a, b]$.

[8] Let $h_n = (b-a)/n$. Denote by

$$T_n = \frac{h_n}{2}(f(a) + 2\sum_{k=1}^{n-1} f(a + kh_n) + f(b))$$

the compound Trapezoidal rule and by

$$S_n = \frac{h_n}{6}(f(a) + 4\sum_{k=1}^{n-1} f(a + (k + \frac{1}{2})h_n) + 2\sum_{k=1}^{n-1} f(a + kh_n) + f(b))$$

the compound Simpson rule. Show that

$$S_n = \frac{4T_{2n} - T_n}{4 - 1}.$$

[9] Let $\triangle := \{x_0, \cdots, x_{n+1}\}$ be a partition of $[a, b]$, i.e, $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$. Let $S_3^2(\triangle)$ be the space of all C^2 cubic spline functions. That is, for any $s \in S_3^2(\triangle)$, $s \in C^2[a, b]$ and $s|_{[x_i, x_{i+1}]}$ is a cubic polynomial, $i = 0, \cdots, n$. For any $f \in C^1[a, b]$, let S_f be the C^2 cubic interpolatory spline of f, i.e., $S_f \in S_3^2(\triangle)$ and

$$S_f(x_i) = f(x_i), i = 0, 1, \cdots, n + 1, S_f'(a) = f'(a), S_f'(b) = f'(b).$$

Suppose that $f \in C^2[a, b]$. Show that

$$\int_a^b |\frac{d^2}{dx^2}(f(x) - S_f(x))|^2 dx \leq \int_a^b |\frac{d^2}{dx^2}(f(x) - s(x))|^2 dx$$

for any $s \in S_3^2(\triangle)$.

[10] Consider solving the following initial value problem of ODE numerically.

$$\begin{cases} y'(x) = f(x, y(x)), & a \leq x \leq b \\ y(a) = \alpha, \end{cases}$$

Derive the Runge-Kutta method of Order 3: $h = (b - a)/n, y_0 = \alpha$ and for $k = 0, \cdots, n - 1$, compute

$$\begin{aligned} K_1 &= f(x_k, y_k) \\ K_2 &= f(x_k + \frac{h}{2}, y_k + \frac{h}{2}K_1) \\ K_3 &= f(x_{k+1}, y_k - hK_1 + 2hK_2) \\ y_{k+1} &= y_k + \frac{h}{6}(K_1 + 4K_2 + K_3). \end{aligned}$$