PROBABILITY THEORY

(Qualifying Examination, 08/16/05)

[Complete Solutions For Any Five Problems Carry Full Credit.]

1. Let X and Y be two random variables. Show that X and Y are independent iff $E[f(X)g(Y)] = E[f(X)]E[g(Y)]$ for any bounded continuous functions f and g.

2. If $\{A_n, n \geq 1\}$ are events satisfying $P(A_n) = o(1)$ and $\sum_{n=1}^{\infty} P(A_n A_{n+1}^c) < \infty$, show that $P(A_n, i.o.) = 0$.

3. Let $X,\ X_n,\ Y^{(\nu)},\ Y_n^{(\nu)}$ be random elements in a metric space (S, ρ) such that $Y_n^{(\nu)} \Longrightarrow Y^{(\nu)}$, as $n \to \infty$, for fixed ν, and moreover $Y^{(\nu)} \Longrightarrow X$, as $\nu \to \infty$, where $\Longrightarrow$ denotes the weak convergence. If, additionally,
$$\lim_{\nu \to \infty} \limsup_{n \to \infty} E\left\{\rho(Y_n^{(\nu)},\ X_n) \wedge 1\right\} = 0$$
holds, show then that $X_n \Longrightarrow X$ as $n \to \infty$.

4. Let $X_n \leq Y_n \leq Z_n$ where $X_n \longrightarrow X,\ Y_n \longrightarrow Y,\ Z_n \longrightarrow Z$ in probability as $n \to \infty$. If $E(X_n) \longrightarrow E(X)$ and $E(Z_n) \longrightarrow E(Z)$, show, using uniform integrability (and NOT Fatou's lemma), that $E(Y_n) \longrightarrow E(Y)$, as $n \to \infty$.

5. (a) Quote, without proof, the Kolmogorov Inequality for independent, zero-mean, L^2-random variables.

   (b) Let $\{X_n\}$ be a sequence of independent random variables such that $\sum_n n^{-2}\mathrm{Var}(X_n) < \infty$. Prove that $\sum_{i=1}^{n} \frac{X_i - E(X_i)}{i} \longrightarrow Y$, *a.s.*, as $n \to \infty$, for some finite random variable Y. Use Kronecker Lemma to deduce that $\frac{1}{n}\sum_{i=1}^{n}(X_i - E(X_i)) \longrightarrow 0$, *a.s.*, as $n \to \infty$.

6. Let $\{X_n,\ n \geq 1\}$ be a sequence of *i.i.d.*, positive, L^1-random variables. Define $T_n := X_1 + \cdots + X_n,\ n \geq 1$, and $N(t) := \max\{n :\ T_n \leq t\}$. Show that $N(t) \longrightarrow \infty$ and that $\frac{N(t)}{t} \longrightarrow \frac{1}{E(X_1)}$, almost surely, as $n \to \infty$.

7. Let X_n and Y_m be independent random variables having the Poisson distribution with parameters n, and m, respectively. Show that

$$\frac{(X_n - n) - (Y_m - m)}{\sqrt{X_n + Y_m}} \Longrightarrow N(0,1), \quad \text{as} \quad n, m \to \infty,$$

where $N(0,1)$ is the standard normal random variable.

8. An urn contains white and black balls. One ball is drawn, and then replaced by two of the same color, and the process is repeated. [Thus, if the urn contains w white and $N - w$ black balls, and a white ball is drawn, the fraction of white balls in the urn before the next drawing is $(w+1)/(N+1)$.] If X_n is the fraction of white balls in the urn before the n-th drawing, show that $\{X_n\}$ is a martingale, and X_n converges almost surely to a limit X_∞, where $E(X_\infty) = E(X_1)$.