UGA High School Varsity Math Tournament
October 25, 2025

WITH SOLUTIONS

Ciphering Round

Time: 2 minutes per problem
Length: 10 problems

Max Score 100 points
10 points for a correct answer.

Problem 1. Assume that $\dfrac{a+b}{a} = 26$ and $\dfrac{b+c}{b} = 82$, what is the value of $\dfrac{c+a}{c} = N$? Write your answer as a simplified fraction.

Answer. $\frac{2026}{2025}$

Solution. We can rewrite the terms as $\dfrac{b}{a} = 26 - 1$, $\dfrac{c}{b} = 82 - 1$ and $\dfrac{a}{c} = N - 1$.

Multiplying these three quantities we find

$$1 = \frac{b}{a} \cdot \frac{c}{b} \cdot \frac{a}{c} = (25)(81)(N-1)$$

so that

$$N = 1 + \frac{1}{25 \cdot 81} = \frac{2026}{2025}$$

Problem 2. Albert (30) and his niece Barbara (17) walk along Milledge. They pass in front of Milan's house and they realize that the house number added to the age of either one is a perfect square. What is Milan's house address?

Answer. 19

Solution. Let h be the house number. Since $17 + h = a^2$ and $30 + h = b^2$, $b^2 - a^2 = 13 = (b - a)(b + a)$. Since 13 is prime, $b - a = 1$ and $b + a = 13$, i.e. $a = 6$ and $b = 7$ and $h = 19$.

Problem 3. What is the length of the longest possible list of distinct elements in $\{1, 2, 3, \ldots, 999\}$ such that each element divides its successor?

Answer. 10

Solution. To maximize the length of the chain, we need the ratio of successive elements to be as small as possible. This can be achieved with a ratio of two and the sequence

$$2^0 = 1, 2^1 = 2, \ldots, 2^9 = 512$$

so that the maximal length is 10.

Problem 4. What is the number of three term geometric progressions

$$0 \leq a_1 \leq a_2 \leq a_3$$

whose middle term is $a_2 = 45$ and all whose terms are integers?

Answer. 8

Solution. For any three term geometric progression $a_1 \leq a_2 \leq a_3$, we must have $a_1 a_3 = a_2^2$. Thus, we are looking for positive integers $a_1 \leq a_3$ such that $a_1 a_3 = 45^2 = 2025$. We see that $2025 = 3^4 5^2$ has $(4+1)(2+1) = 15$ divisors, so the answer is 8.

Problem 5. Let $p(x)$ be a polynomial such that

$$p(3x - 2) = 9x^2 + 6x + 1.$$

What is the smallest value v for which $p(v) = 0$?

Answer. -3

Solution. The right hand side of the expression equals $(3x + 1)^2 = ((3x - 2) + 3)^2$ so that $p(v) = (v + 3)^2$ whose only root is -3.

Problem 6. What is the value of the fraction

$$\frac{1+3+5+\cdots+2023+2025}{2027+2029+\cdots+4049+4051},$$

where the entries of both numerator and denominator are in arithmetic progression.

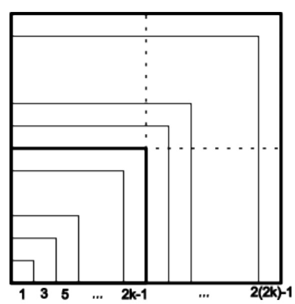
Answer. $\frac{1}{3}$

Solution.

1. **Engineer solution**. Truncating the expression we notice that

$$\frac{1}{3}=\frac{1+3}{5+7}=\frac{1+3+5}{7+9+11}=\ldots=\frac{1}{3}.$$

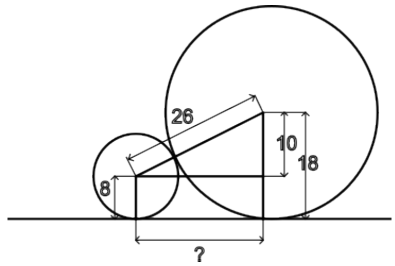
2. **Sum of odd numbers.** The sum of the first k odd numbers is k^2 (as can be seen from the picture below). Similarly, the denominator is the difference $(2k)^2-(k)^2=3k^2$. Thus the ratio is $\frac{k^2}{3k^2}=\frac{1}{3}$. Again, this can be seen from the picture below, the numerator is the bottom left corner, the numerator, the remainder of the large square.



Problem 7. Two circles, C_1 and C_2, with respective radii of 8 and 18, are tangent at a point A. A common tangent line, ℓ, touches the circles at distinct points B and C. Find the length of the segment BC.

Answer. 24

Solution. Using the fact that the radii of circles meeting at the point of tangency are aligned and that radii are perpendicular to the tangent line they intersect we get the picture below.



One deduces then that the length of BC is $\sqrt{26^2 - 10^2} = 24$.

Problem 8. What is the number of three term arithmetic progressions

$$1 \leq a_1 \leq a_2 \leq a_3$$

all whose terms are elements of the set $\{1, 2, \ldots, 99, 100\}$?

Answer. 2550

Solution. For any three term arithmetic progression $a_1 \leq a_2 \leq a_3$, we must have $\frac{a_1+a_3}{2} = a_2$. Thus, if a_1, a_2, a_3 are all integers, then it suffices to find all integers $1 \leq a_1 \leq a_3 \leq 100$ such that a_1 and a_3 have the same parity. There are $2 \cdot \binom{50}{2} + 100 = 2550$ such pairs. (We added 100 for the degenerate progressions of the form a, a, a.)

Problem 9. Jenny has a funny way of counting till 10. She always starts with 1. However, instead of simply calling out 2, she randomly calls out a number from 2 to 10 with equal probability. She continues in this manner, randomly calling out one of the numbers she has not called out with equal probability, until she calls out 10, at which point she stops. For example, one day she might count

$$1, 7, 2, 10.$$

What is the expected value of the sum of the numbers that she will end up saying out loud?

Answer. 33

Solution. Intuitively, each number outside of 1 and 10 would have a 50% chance of being called; this suggests that by linearity of expectation the expected sum would be $1 + 10 + 0.5(2 + 3 + \cdots + 9) = 33$. This turns out to indeed be the correct answer, though due to the role of order in this count, this 50% chance is less obvious[1]. Here is a more rigorous approach.

[1] This can be made rigorous by considering 10 as being the border between numbers spoken out and those not spoken out. The symmetry is now clear.

We note that there is a $1/9$ chance that 10 immediately gets called out after 1, and that the count ends right away, having 2 terms. There is then an $8/9$ chance that the count does not end immediately, and a $1/8$ chance that 10 is called out as the third number, ending the count; there is thus a $8/9 \cdot 1/8 = 1/9$ chance that the count has 3 terms. Proceeding similarly for each $k = 2, 3, \ldots, 10$, there is a $1/9$ chance that Jenny calls out exactly k numbers.

We now look at the expected value of the sum of all numbers in a count of exactly k terms. The stipulated conditions give us that each possible count appears with equal probability. Pairing up each count $1, a_2, a_3, \ldots, a_{k-1}, 10$ with the corresponding sequence $1, 11 - a_2, 11 - a_3, \ldots, 10$ (which notably is a distinct sequence) then tells us that the expected sum of terms of a length-k count is $5.5k$. Averaging over $k = 2, 3, \ldots 10$ gives us an expected sum of $5.5(2 + 3 + \cdots + 10)/9 = 33$.

Problem 10. Let x be a positive integer. In base nine, the sum of its digit is 8; in base six, the sum of its digits is 5. What is the smallest possible value of x?

Answer. 40

Solution. For any integer x written in base b, x is congruent to the sum of its digits in base b, modulo $b - 1$. The archetype of this property is that an integer is congruent modulo 9 to the sum of its digits in the usual base 10 notation.

Hence,

$$x \equiv 8 \pmod{8} \quad \Rightarrow \quad x \equiv 0 \pmod{8},$$

and

$$x \equiv 5 \pmod{5} \quad \Rightarrow \quad x \equiv 0 \pmod{5}.$$

Thus x is divisible by both 8 and 5, so x is a multiple of 40 which is the smallest possible value.